Path integrals for boundaries and topological constraints: A white noise functional approach

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Using the Streit–Hida formulation where the Feynman path integral is realized in the framework of white noise analysis, we evaluate the quantum propagator for systems with boundaries and topological constraints. In particular, the Feynman integrand is given as generalized white noise functionals for systems with flat wall boundaries and periodic constraints. Under a suitable Gauss–Fourier transform of these functionals the quantum propagator is obtained for: (a) the infinite wall potential; (b) a particle in a box; (c) a particle constrained to move in a circle; and (d) the Aharonov–Bohm system. The energy spectrum and eigenfunctions are obtained in all four cases. © 2002 American Institute of Physics.

I. INTRODUCTION

In 1983, Streit and Hida used white noise analysis to provide a more rigorous basis for the Feynman path integral. This approach, where the Feynman integrand is identified as a generalized white noise functional, was subsequently developed and applied to various quantum mechanical systems including time-dependent potentials. In this paper, we shall employ the same white noise functional approach to evaluate the Feynman path integral for systems with constraints brought about by specified boundary and topological conditions. We begin with a brief review of the white noise functional approach in Sec. II. The free particle propagator is then evaluated as an example and basic tool in the treatment of the constrained systems. Section III presents the evaluation of the propagator for flat wall boundaries such as a quantum particle in a region bounded by an infinite wall. Then the Feynman integral for a particle in a one-dimensional box is examined, where we employ the Poisson sum formula to evaluate the propagator. Section IV considers systems with periodic constraints exemplified by the quantum particle constrained to move in a circle. Here we again utilize the Poisson sum formula instead of introducing smeared wave packets as done in Ref. 16, where a white noise approach was also used. In the Aharonov–Bohm setup, a charged particle is constrained to move in a circle around a solenoid containing a magnetic flux situated at the center. We obtain the propagator for this system which readily yields the Aharonov–Bohm energy spectrum and eigenfunctions. In the limit where the flux vanishes, one obtains the expected result for a quantum particle in a circle.

II. BRIEF REVIEW

A. The Feynman path integral

The quantum mechanical propagator, \( K(x_1, t | x_0, 0) = \exp(-iHt)\delta(x_1 - x_0) \), in nonrelativistic quantum mechanics can be calculated using Feynman’s prescription of summing over all possible paths or “histories” of the particle which start at \( x_0 \) and end at \( x_1 \). This prescription is symbolically written as

\[ \langle x_1 | \exp(-iHt) | x_0 \rangle = K(x_1, t | x_0, 0) \]

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K(x_1,t|x_0,0) = \int \exp\left(\frac{i}{\hbar} S\right) D[x], \quad (2.1)

where S is the classical action for a particle of mass \( \mu \) subjected to a potential \( V(x) \) given by

\[
S = \int_0^t \left[ \frac{1}{2} \mu \dot{x}^2 - V(x) \right] dt.
\]

A useful procedure in the explicit calculation of the path integral equation (2.1) is to slice the time interval \( N \) times, i.e., \( t/N = t_j - t_{j-1} = \epsilon_j (j = 1, \ldots, N) \), to obtain the time-sliced form,

\[
K(x_1,t|x_0,0) = \lim_{N \to \infty} \int \prod_{j=1}^N A_j \exp\left(\frac{i}{\hbar} S_j\right) \prod_{j=1}^{N-1} d[x_j], \quad (2.3)
\]

where, \( x_j = x(t_j) \), \( S_j \) is the short-time action, and \( A_j = \sqrt{\mu/2\pi i \hbar} \epsilon_j \) is the normalization. Although numerous quantum mechanical problems are solved using this prescription of Feynman, there is still continuing effort to investigate the mathematical meaning of the Feynman path integral. Specifically, the integral, Eq. (2.1) or Eq. (2.3), with its infinite-dimensional flat “measure” \( D[x] \) or \( \Pi d[x_j] \) is not mathematically well defined. There are several approaches for providing the Feynman integral with a more solid mathematical foundation, but what we shall follow in this paper is the approach of Streit and Hida, which utilizes the infinite-dimensional white noise calculus.

**B. White noise analysis and the Feynman integral**

What is referred to as white noise \( \omega(t) \) is a random process defined as the time derivative of the Brownian motion \( B(t) \), i.e.,

\[
\omega(t) = dB(t)/dt.
\]

Therefore, the white noise \( \omega(t) \) may be viewed as the “velocity” of a Brownian motion. Alternatively, Wiener’s Brownian motion \( B(t) \) is represented by

\[
B(t) = \int_0^t \omega(\tau) d\tau.
\]

White noise calculus was introduced by Hida in 1975 as a novel approach to infinite dimensional analysis. The basic idea was to take the collection of infinitely many independent random variables, \( \{\omega(t); t \in \mathbb{R}\} \), and treat them as the coordinate system of an infinite dimensional space. One then proceeds to investigate generalized white noise functionals, \( \Phi(\omega(t); t \in \mathbb{R}) \), instead of functionals of Brownian motion, \( f(B(t); t \in \mathbb{R}) \).

In the framework of white noise analysis, the Feynman integral is treated as the “average over all paths” with a well-defined generalized white noise functional, or Hida distribution, as the weight. The paths \( x(t) \) which start at \( x_0 \) are parametrized as

\[
x(t) = x_0 + \left( \frac{\hbar}{\mu} \right)^{1/2} \int_0^t \omega(\tau) d\tau,
\]

in terms of the white noise random variable \( \omega(t) \). Here, \( \mu \) is the mass of the particle. With this, the velocity of the particle becomes \( (dx/dt) = \sqrt{\hbar/\mu} \omega \), enabling us to write the exponential of \( (i\hbar)S_0 \), where \( S_0 \) is the action for the free particle, as

\[
\exp\left(\frac{i}{\hbar} S_0\right) = \exp\left[ \frac{i}{\hbar} \int_0^t \frac{1}{2} \mu \dot{x}^2 \right] = \exp\left[ \frac{i}{2} \int_0^t \omega(\tau)^2 d\tau \right]. \quad (2.7)
\]

The next step is to interpret the Feynman integration over all paths, \( \lim_{N \to \infty} \Pi d[x_j] \) or \( d^\infty x \), in terms of integration over the Gaussian white noise measure \( d\mu(\omega) \).
\[ d\mu(\omega) = N_\omega \exp\left( -\frac{1}{2} \int \omega(\tau)^2 \, d\tau \right) \, d\omega. \] (2.8)

We observe that, in the framework of white noise analysis, Feynman’s flat “measure” \(N^0 dx\) would correspond to \(N_\omega \, d\omega = \exp[(1/2)\int_0^t \omega(\tau)^2 \, d\tau] \, d\mu(\omega)\). With this, we also get the correspondence (\(\hbar = 1\)),

\[ \exp(iS_0)D[x] \rightarrow N \exp\left[ \frac{i+1}{2} \int_0^t \omega(\tau)^2 \, d\tau \right] \, d\mu(\omega), \] (2.9)

where \(N\) is a suitable normalization factor.

The paths in the Feynman integral begin at \(x_0\) and end at \(x_1\). However, the parametrization of the paths \(x(t)\) in Eq. (2.6) shows that only the initial point \(x_0\) is fixed from where the random Brownian motion begins. We, therefore, fix the end point of the trajectories by means of the Donsker delta function,\(^2,16\) \(\delta(x(t) - x_1)\), where \(x(t)\) is given by Eq. (2.6), such that at time \(t\) the particle is at the final point \(x_1\). The Feynman integrand can now be represented by

\[ I(x_1, t|x_0, 0) = N \exp\left( \frac{i+1}{2} \int_0^t \omega(\tau)^2 \, d\tau \right) \delta(x_0 + (\hbar/\mu)^{1/2} \int_0^t \omega(\tau) \, d\tau - x_1), \] (2.10)

in terms of the Gauss kernel,

\[ I_0 = N \exp\left( \frac{i+1}{2} \int_0^t \omega(\tau)^2 \, d\tau \right). \] (2.11)

Equipped with the Feynman integrand as a generalized white noise functional, and the measure in white noise space, the path integral is obtained by performing a “\(T\)-transform” of the Feynman integrand. Explicitly, this is defined as\(^2-4\)

\[ (T\Phi)(\xi) = \int \exp\left( i \int \omega(\tau) \xi(\tau) \, d\tau \right) \Phi(\omega) \, d\mu(\omega), \] (2.12)

with \(\xi \in \mathcal{S}\), \(\Phi \in \mathcal{S}^*\), for the triple \(\mathcal{S} \subset L^2(\mu) \subset \mathcal{S}^*\), the \(\mathcal{S}^*\) being the white noise measure space. Here \(d\mu(\omega)\) is the Gaussian white noise measure\(^2-4\) characterized by its Fourier transform,

\[ \int \exp(i\langle \omega, \xi \rangle) \, d\mu(\omega) = \exp\left( -\frac{1}{2} \int \xi^2 \, d\tau \right) = C(\xi), \] (2.13)

where \(C(\xi)\) is the characteristic functional. For example, the \(T\)-transform of \(I_0\), Eq. (2.11), is\(^2,6\)

\[ (TI_0)(\xi) = \exp\left( -\frac{i}{2} \int \xi^2 \, d\tau \right). \] (2.14)

Similarly, the \(T\)-transform of the functional \(I(x_1, t|x_0, 0)\), Eq. (2.10), is given by (\(\hbar = \mu = 1\))

\[ (TI)(\xi) = \frac{1}{(2\pi i)^{1/2}} \exp\left[ -\frac{i}{2} \int_0^t \xi^2(\tau) \, d\tau \right] \times \exp\left[ \frac{i}{(2\pi)} \left( \int_0^t \xi(\tau) \, d\tau + x_1 - x_0 \right)^2 \right]. \] (2.15)

The \(I(x_1, t|x_0, 0)\), identified as the Feynman integrand, exists as a Hida distribution.\(^2,6\) For \(\xi = 0\), this yields the free particle propagator,

\[ K(x_1, t|x_0, 0) = (TI)(0) = \frac{1}{(2\pi i)^{1/2}} \exp\left[ \frac{i}{(2\pi)} (x_1 - x_0)^2 \right]. \] (2.16)

We shall now employ this white noise approach to problems with boundaries and systems involving topologically inequivalent paths.
III. FLAT WALL BOUNDARIES

A. Infinite wall

We first consider a quantum particle of mass $m$ in a potential which describes an infinite wall at the origin, i.e.,

$$V(x) = \begin{cases} \infty & \text{for } x \leq 0 \\ 0 & \text{for } x > 0. \end{cases} \quad (3.1)$$

Classically, for a particle which goes from $x_0$ to $x_1$ there are two possible paths: the first is a path that goes directly from the initial to the final point, and the second describes a path from $x_0$ that is reflected by the wall before arriving at $x_1$. Quantum mechanically, the particle propagator should satisfy the boundary condition,

$$K(x_1, x_0) = 0$$

at $x_1 = 0$ or $x_0 = 0$, \quad (3.2)

together with

$$\lim_{t \to 0} K(x_1, x_0; t) = \delta(x_1 - x_0). \quad (3.3)$$

To obtain the propagator for the infinite wall problem we use the path parametrization of Eq. (2.6) and write the linear combination of white noise functionals,

$$I^W(x_1, t|x_0, 0) = I_0 \left[ x_0 + (1/\mu)^{1/2} \int_0^t \omega(\tau) d\tau - x_1 \right] - I_0 \left[ -x_0 + (1/\mu)^{1/2} \int_0^t \omega(\tau) d\tau - x_1 \right], \quad (3.4)$$

where $I_0$ is the Gauss kernel given by Eq. (2.11). For a propagator which satisfies the boundary condition, Eq. (3.2), the combination of the type given by Eq. (3.4) has been discussed in the literature, \cite{15,20,21} where the second term arises from particle trajectories originating from an image point, $-x_0$, and arriving at $x_1$.

Writing the delta function in terms of its Fourier representation we express the functional in Eq. (3.4) as ($\hbar = 1$)

$$I^W(x_1, t|x_0, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\exp[i(kx_0 - x_1)] - \exp[-ik(x_0 + x_1)]\} \times I_0 \exp \left[ (ik/\sqrt{\mu}) \int_0^t \omega(\tau) d\tau \right] dk. \quad (3.5)$$

We then perform the $T$-transform of Eq. (3.5) following the definition in Eq. (2.12),

$$\left( T I^W \right)(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\exp[i(kx_0 - x_1)] - \exp[-ik(x_0 + x_1)]\} \left[ I_0 \left( \frac{\xi + k/\sqrt{\mu}}{\mu} \right) \right] dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\exp[i(kx_0 - x_1)] - \exp[-ik(x_0 + x_1)]\} \exp \left[ -i \frac{1}{2} \int_0^t \left[ \xi + (k/\sqrt{\mu}) \right] d\tau \right] dk. \quad (3.6)$$

With $\xi = 0$, the usual quantum propagator $(TI^W(0)) = K^W(x_1, t|x_0, 0)$ can be obtained:
describes a path which bounces once from the boundary at \( x \) for a particle in a box can be written as

\[
K(x, t | x_0, 0) = \int_{-\infty}^{+\infty} \Psi_k(x_0) \Psi_k(x_1) \exp(-iE_k t) dk, \tag{3.7}
\]

where the energy is, \( E_k = (k^2/2\mu) \), and the eigenfunction is given by \( \Psi_k(x) = (1/\sqrt{\pi})\sin(kx) \).

**B. Particle in a box**

We next consider the Feynman integral for a particle of mass \( \mu \) confined to move in a one-dimensional box of length \( L \), with sides located at \( x = 0 \) and \( x = L \). Classically, the paths of the particle in a box can be categorized into four classes.\(^{15}\) The first goes directly from \( x_0 \) to \( x_1 \) without hitting the walls; the second type of path hits the wall at \( x = 0 \) before arriving at \( x_1 \); the third leaves \( x_0 \) and is reflected from the wall at \( x = L \) before reaching \( x_1 \); and the fourth describes a path which bounces once from the boundary at \( x = 0 \) and also once from the wall at \( x = L \) before reaching \( x_1 \). The other paths belonging to any of the four classes describe a particle bouncing back and forth inside the box of length \( L \) and, therefore, traveling an additional distance of \( 2Ln(n = 0, 1, 2, \ldots) \).

Quantum mechanically, the particle propagator \( K(x_1, x_0 ; t) \) has to satisfy the boundary conditions

\[
K(x_1, 0 ; t) = K(x_1, L ; t) = K(0, x_0 ; t) = K(L, x_0 ; t) = 0, \tag{3.8}
\]

aside from the requirement

\[
\lim_{t \to 0} K(x_1, x_0 ; t) = \delta(x_1 - x_0). \tag{3.9}
\]

Using the form of the Feynman functional for a free particle, Eq. (2.10), the white noise functional for a particle in a box can be written as

\[
I^B(x_1, t | x_0, 0) = \sum_{n=-\infty}^{+\infty} I_0 C_n(x_1, t | x_0, 0), \tag{3.10}
\]

where\(^{14}\)

\[2C_n(x_1, t | x_0, 0) = \delta(x(t) - x_1 + 2Ln) + \delta(-x(t) + x_1 + 2Ln) - \delta(x(t) + x_1 + 2Ln) \]

\[\delta(-x(t) - x_1 + 2Ln) \]  \tag{3.11}

with \( x(t) \) given by Eq. (2.6). The \( (2Ln) \) in the delta function describes the fact that the particle leaving \( x_0 \) can bounce back and forth inside the box of length \( L \) before arriving at \( x_1 \). In Eq. (3.11) we observe that the first two terms, as well as the last two terms can be combined, respectively, so that Eq (3.10) can be written as

\[
I^B(x_1, t | x_0, 0) = \sum_{n=-\infty}^{+\infty} I_0 \left\{ \delta\left(x_0 + (1/\mu)^{1/2} \int_0^t \omega(\tau) d\tau - x_1 + 2Ln\right) \right. 
\]

\[\left. - \delta\left(x_0 + (1/\mu)^{1/2} \int_0^t \omega(\tau) d\tau + x_1 + 2Ln\right) \right\}. \tag{3.12}
\]

We now write \( \delta(x_1 - x_1 + 2Ln) = (\pi/L) \delta((\pi/L)[x(t) - x_1] + 2\pi n) \), where \( x(t) \) is given by Eq. (2.6). We can then apply the Poisson sum formula,\(^{22}\)

\[
\sum_{n=-\infty}^{\infty} \delta(\phi + 2\pi n) = (1/2\pi) \sum_{m=-\infty}^{\infty} \exp(i m \phi), \tag{3.13}
\]
and write Eq. (3.12) as

\[
I^B(x_1, t|x_0, 0) = (1/2L) \sum_{m = -\infty}^{+\infty} I_0 \exp \left[ i m \pi L \sqrt{\mu} \int_0^{t} \omega(\tau) \, d\tau \right]
\times \{ \exp[(i m \pi/L)(x_0 - x_1)] - \exp[(i m \pi/L)(x_0 + x_1)] \}. \tag{3.14}
\]

Taking the \( T \)-transform of the functional in Eq. (3.14) we get

\[
(TI^B)(\xi) = (1/2L) \sum_{m = -\infty}^{+\infty} (TI_0)(\xi + (m \pi/L \sqrt{\mu}))
\times \{ \exp[(i m \pi/L)(x_0 - x_1)] - \exp[(i m \pi/L)(x_0 + x_1)] \}
= (1/2L) \sum_{m = -\infty}^{+\infty} \exp \left[ - \frac{i}{2} \int_0^{t} \left( \xi + \frac{m \pi}{L \sqrt{\mu}} \right)^2 \, d\tau \right]
\times \{ \exp[(i m \pi/L)(x_0 - x_1)] - \exp[(i m \pi/L)(x_0 + x_1)] \}. \tag{3.15}
\]

Taking \( \xi = 0 \), we obtain the propagator,

\[
K^B(x_1, t|x_0, 0) = (TI^B)(0) = (1/2L) \sum_{m = -\infty}^{+\infty} \exp[-(i/2)(m \pi/L \sqrt{\mu})^2 t]
\times \{ \exp[(i m \pi/L)(x_0 - x_1)] - \exp[(i m \pi/L)(x_0 + x_1)] \}
= (1/2L) \sum_{m = -\infty}^{+\infty} \exp[-(i/2)(m \pi/L \sqrt{\mu})^2 t]
\times \left[ \cos \left( \frac{m \pi}{L} (x_0 - x_1) \right) - \cos \left( \frac{m \pi}{L} (x_0 + x_1) \right) \right], \tag{3.16}
\]

with the sum over terms such as \( \sin((m \pi/L)(x_0 - x_1)) \) giving a zero contribution. Equation (3.16) can be written in the symmetrized form,

\[
K^R(x_1, t|x_0, 0) = \sum_{m = -\infty}^{+\infty} \phi_m(x_0) \phi_m(x_1) \exp(-iE_mt), \quad \tag{3.17}
\]

where \( E_m = m^2 \pi^2/2\mu L^2 \) and \( \phi_m(x) = (1/\sqrt{L}) \sin(m \pi x/L) \) are the energy eigenvalues and eigenfunctions for a quantum particle in a one-dimensional box.

IV. SYSTEMS WITH TOPOLOGICAL CONSTRAINTS

A. Quantum particle in a circle

In this section, we consider white noise functionals for a quantum system in a space which is multiply connected.\cite{15,23,25} In particular, we treat the case of a particle constrained to move in a circle. This has been examined in the context of white noise analysis using smeared wave packets in Ref. 16. In contrast to this earlier work, we shall employ the Poisson sum formula, Eq. (3.13), to facilitate the evaluation of the propagator.

Let us begin by expressing the paths of the particle in a circle as \( (\hbar = 1; \ t_0 = 0) \),

\[
\theta(t) = \theta_0 + \frac{1}{\sqrt{L}} \int_0^{t} \omega(\tau) \, d\tau, \quad \tag{4.1}
\]
where \( I = \mu R^2 \), for a particle of mass \( \mu \) and a circle of radius \( R \). We then note that the particle starting at a point \( \theta_0 \) can move clockwise, or counterclockwise, before reaching a final point \( \theta_1 \). The particle may even wind around the circle \( n \) times counterclockwise (for \( n \) positive), or \( |n| + 1 \) times clockwise (for \( n \) negative) before stopping at \( \theta_1 \). In contrast to Eq. (2.10) for the free particle, we must sum over all possible paths with different winding numbers and utilize the winding number decomposition of the propagator\(^{15,23,24}\) to write

\[
I^C(\theta_1, t | \theta_0, 0) = \sum_{n = -\infty}^{+\infty} I_0 \exp \left[ im \left( \theta_0 + \frac{1}{\sqrt{I}} \int_0^t \omega(\tau) \, d\tau - \frac{1}{2} \omega(\tau) \, d\tau \right) \right],
\]

(4.2)

where \( n \) is the winding number, and the kinetic part \( I_0 \) is given by Eq. (2.11).

Use of the Poisson sum formula, Eq. (3.13), in the functional Eq. (4.2) gives

\[
I^C(\theta_1, t | \theta_0, 0) = (1/2\pi) \sum_{m = -\infty}^{+\infty} I_0 \exp \left[ i m \left( \theta_0 + \frac{1}{\sqrt{I}} \int_0^t \omega(\tau) \, d\tau - \theta_1 \right) \right].
\]

(4.3)

Taking the \( T \)-transform we get

\[
(TI^C)(\xi) = (1/2\pi) \sum_{m = -\infty}^{+\infty} \exp \left[ i m(\theta_0 - \theta_1) \right] \left[ TI_0 \exp \left( i m/\sqrt{I} \right) \int_0^t \omega(\tau) \, d\tau \right](\xi)
\]

\[
= (1/2\pi) \sum_{m = -\infty}^{+\infty} \exp \left[ i m(\theta_0 - \theta_1) \right] \left[ (TI_0)(\xi + (m/\sqrt{I})) \right]
\]

\[
= (1/2\pi) \exp \left( -\frac{i}{2} \int \xi^2 \, d\tau \right) \sum_{m = -\infty}^{+\infty} \exp \left[ i m(\theta_0 - \theta_1) \right] \exp \left( -\frac{im^2 \tau}{2I} - \frac{im}{\sqrt{I}} \int_0^t \xi(\tau) \, d\tau \right).
\]

(4.4)

For \( \xi = 0 \), this yields the propagator for a charged particle constrained to move in a circle,

\[
K(\theta_1, t | \theta_0, 0) = (TI^C)(0) = (1/2\pi) \sum_{m = -\infty}^{+\infty} \exp \left[ i m(\theta_0 - \theta_1) - iE_m t \right],
\]

(4.5)

where the energy is \( E_m = m^2/2\mu R^2 \) in terms of the angular quantum number, \( m \).

**B. The Aharonov–Bohm set up**

The Aharonov–Bohm setup\(^{26–28}\) can be described by a charged particle which moves in a space with an impenetrable solenoid of radius \( R = 0 \) that carries a flux \( \Phi \). For a solenoid oriented along the \( z \) axis, we can make use of the symmetry of the problem and look at the cross section, or the \((x-y)\) plane, where the Lagrangian for the particle is given by

\[
L = \frac{1}{2} \mu \dot{r}^2 + e \mathbf{A} \cdot \dot{r},
\]

(6.6)

Here, \( \mu \) is the mass of the particle, \( \dot{r} = (d\mathbf{r}/dt) \), and the vector potential is given by

\[
\mathbf{A} = \Phi \left( \frac{-y\hat{i} + x\hat{j}}{2\pi \mathbf{r}^2} \right) \mathbf{r},
\]

\[ x^2 + y^2 > R^2 \]

(4.7)

such that the magnetic field is, \( \mathbf{B} = \nabla \times \mathbf{A} = 0 \); outside the solenoid. In polar coordinates, \( \mathbf{r} = (r, \theta) \), the potential in the Lagrangian has the form.
\[ \frac{e}{c} \vec{A} \cdot \vec{v} = \frac{e \Phi}{2 \pi c} \dot{\theta}. \] (4.8)

By constraining the particle to move in a circle with the solenoid at the center and located at the origin, the Lagrangian, Eq. (4.6), acquires the following form:

\[ L = \frac{1}{2} I \dot{\theta}^2 + \frac{e \Phi}{2 \pi c} \dot{\theta}, \] (4.9)

where, \( I = \mu R^2 \). The problem can now be cast in the language of white noise by modeling the paths of the particle using Eq. (4.1), such that Eq. (4.9) can be written as

\[ L = \frac{1}{2} \omega^2 + \frac{e \Phi}{2 \pi c \sqrt{I}} \omega. \] (4.10)

As in the case of a particle in a circle discussed previously, we can again use the winding number decomposition of the propagator, with \( n \) being the winding number, and use the Lagrangian, Eq. (4.10), for the Aharonov–Bohm setup to write

\[ I^{AB}(\theta_1, t|\theta_0, 0) = \sum_{n=-\infty}^{+\infty} I_0 \left[ \theta_0 + \frac{1}{\sqrt{I}} \int_0^t \omega(\tau) d\tau - \theta_1 + 2\pi n \right] \exp \left[ i \frac{\alpha}{\sqrt{I}} \int_0^t \omega(\tau) d\tau \right], \] (4.11)

where \( I_0 \) is given by Eq. (2.11) and the last factor contains the potential term with the magnetic flux, \( \alpha = e\Phi/2\pi c \). Using the Poisson sum formula, Eq. (3.13), the functional, Eq. (4.11), becomes

\[ I^{AB}(\theta_1, t|\theta_0, 0) = (1/2\pi) \sum_{m=-\infty}^{+\infty} I_0 \exp[im(\theta_0 - \theta_1)] \exp \left[ i \frac{(m + \alpha)}{\sqrt{I}} \int_0^t \omega(\tau) d\tau \right], \] (4.12)

whose \( T \)-transform gives us

\[ (TI^{AB})(\xi) = (1/2\pi) \sum_{m=-\infty}^{+\infty} \exp[im(\theta_0 - \theta_1)] \left( T \left[ I_0 \exp \left( i \frac{(m + \alpha)}{\sqrt{I}} \int_0^t \omega(\tau) d\tau \right) \right] \right)(\xi) \]
\[ = (1/2\pi) \sum_{m=-\infty}^{+\infty} \exp[im(\theta_0 - \theta_1)] \left( TI_0 \left( \xi + \frac{(m + \alpha)}{\sqrt{I}} \right) \right) \]
\[ = (1/2\pi) \exp \left( -\frac{i}{2} \int \xi^2 d\tau \right) \sum_{m=-\infty}^{+\infty} \exp[im(\theta_0 - \theta_1)] \]
\[ \times \exp \left[ -i \frac{(m + \alpha)^2 t}{2I} - i \frac{(m + \alpha)}{\sqrt{I}} \int_0^t \dot{\xi}(\tau) d\tau \right]. \] (4.13)

For \( \xi = 0 \), this yields the Aharonov–Bohm propagator for a charged particle constrained to move in a circle,

\[ K^{AB}(\theta_1, t|\theta_0, 0) = (TI^{AB})(0) = (1/2\pi) \sum_{m=-\infty}^{+\infty} \exp[im(\theta_0 - \theta_1) - iE_m t], \] (4.14)

where the energy is \( E_m = [m + (e\Phi/2\pi c)]^2/2\mu R^2 \), with \( m \) the angular quantum number modified by the magnetic flux \( e\Phi/2\pi c \). For the case where the magnetic flux \( \Phi = 0 \), the Aharonov–Bohm propagator reduces to that of the particle in a circle discussed in the previous section.
V. CONCLUSION

In this paper, Feynman integrals for some systems with boundaries or constraints have been explicitly evaluated using the white noise functional approach. This approach was first introduced by Streit and Hida in Ref. 1, where they constructed the Feynman integral in the framework of white noise analysis. This infinite dimensional calculus provides a natural setting for the “sum over all trajectories” while quantum analysis is done in real time without resort to time slicing. Several quantum systems have been treated explicitly using this method,7–12 and this paper adds topologically constrained systems to the list. These solvable examples should shed more light on the true mathematical meaning of the Feynman integral, as well as pave the way for the treatment of more complicated systems with boundaries and nontrivial non-Gaussian systems.

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