Unimodality of price-setting newsvendor’s objective function with multiplicative demand and its applications

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Abstract
We present a general solution framework for the price-setting newsvendor problem with a multiplicative stochastic demand. Under mild assumptions, such as increasing price elasticity on the mean demand function and increasing generalized failure rate on the distribution of the random factor, we first prove that both the profit function with respect to price and its derived function with respect to order quantity are quasi-concave. Three applications are then studied under our solution framework: (1) We consider a wholesale price only contract by which a manufacturer sets a wholesale price and a newsvendor determines an order quantity and the retail price, and show that the manufacturer’s profit function is unimodal with respect to retailing price or stocking factor under certain conditions. (2) We consider a newsvendor problem in which the demand depends on both the retail price and the level of sales effort, and the cost exerting the sales effort is proportional to the order quantity; we prove that there exists a unique pair of price and sales-effort levels that maximize the total profit. This result is established under a set of mild assumptions on the demand and cost functions. (3) We identify a property in the single-period profit function that satisfies Condition 1 of Huh and Janakiraman (2008), which in turn guarantees the optimality of (s, S) policy for an infinite stationary dynamic inventory-price control system with lost sales and fixed order costs. Finally, the unimodality of the newsvendor problem with a general stochastic and price-sensitive demand is studied.

1. Introduction

Cross-functional decisions, such as joint pricing and inventory control, have received much attention in research areas ranging from supply-chain coordination to multi-period inventory management. A building block of these models is the price-setting newsvendor problem. Petruzzi and Dada (1999) provide a review on and extensions to the price-sensitive newsvendor problem with either an additive or a multiplicative random term in the demand function. A basic question for the price-setting newsvendor problem is the existence and uniqueness of optimal price and order quantity decisions. The primary purpose of this paper is to study the unimodality of the price-setting newsvendor’s objective function with multiplicative demand, an appealing property for the study of uniqueness of optimal solution.

The multiplicative demand function of the newsvendor consists of two parts: a mean demand that decreases in price and a random factor that is independent of price. Instead of reviewing the vast literature on joint pricing and inventory control, in what follows, we attempt to position our paper with respect to similar research.

To guarantee the unimodality of the objective function and hence the uniqueness of optimal solution, most of the existing literature on the price-setting newsvendor problem confines either the mean demand or the distribution of random factor to specific functional forms. For a comprehensive survey, see Yao et al. (2006). Zabel (1970) studies the uniqueness of the optimal price of the price-setting newsvendor problem with a multiplicative demand by explicitly assuming the concavity of expected revenue, which is exactly the mean demand multiplied by price. Polatoglu (1991) considers a general newsvendor model with a price-sensitive demand and gives a sufficient condition for the profit function to be unimodal, which however is difficult to verify in general. When the demand is multiplicative, a closed-form condition is provided for the case with a linear mean demand and an exponential distribution of the random factor. By assuming that the distribution of the random factor satisfies some restrictive conditions that encompass the increasing failure rate, Petruzzi and Dada (1999) investigate the uniqueness of the optimal solution to the newsvendor problem with the mean
demand of power function and multiplicative randomness. Under specific, such as iso-price-elastic and linear multiplicative, demand settings, Wang et al. (2004) provide sufficient conditions for the unimodality of the profit functions and the uniqueness of optimal price and order quantity. Under a price-setting newsvendor framework, Song et al. (2008) consider the wholesale price only contract with buybacks. They develop the concavity of the newsvendor’s objective function in the order quantity. In comparison with our conditions, their demand function is general in terms of the distribution of random component but restrictive for the mean demand component.

Instead of directly investigating the properties on the demand function, Kocabiyikoglu and Popescu (2011) associate the concavity and submodularity of the newsvendor’s profit function with the properties of the so-called elasticity of lost-sales rate under a general demand model setting. The uniqueness of optimal price–quantity decision depends on the concavity of riskless unconstrained revenue, a stronger condition than the increasing price elasticity assumed in our setting when the demand function is of a multiplicative form (refer to Lemma 2 and Theorem 1). When the demand function is of a general form, the sufficient conditions to guarantee the uniqueness of optimal solution differs from those of ours (see Section 6). In a most recent paper, Arikan and Jammernegg (2009) focus on the distribution functions of demand depending on price. In particular, as price increases probability of low demand increases and consequently expected demand decreases, and the demand is not required to be separated into a price dependent deterministic term and a price independent random term. They introduce the so-called price elasticity of expected sales and show that unimodality is assured as long as such an elasticity is increasing in price. However, it remains a challenge to verify whether a demand function has an increasing price elasticity of sales.

In Section 2 we formally introduce the basic newsvendor model and study the unimodality or quasi-concavity of the price-setting newsvendor’s objective function. Our interest is to identify relatively general conditions such as increasing general failure rate (IGFR) on the random factor and increasing price elasticity (IPE) on the mean demand, either of which has broad applications in operations management, as will be illustrated in the next section. Our basic model is also closely related to the following. In the case with deterministic price-sensitive demand, Rajan et al. (1992) study the uniqueness of optimal price under the convexity of the reciprocal of the demand function, a property less restrictive than IPE. Ziya et al. (2004) discuss the relationships among three common assumptions that ensure a well-behaved revenue function, the second of which actually corresponds to our IGFR assumption. In considering a non-profit organization’s investment decision, de Vericourt and Lobo (2009) show the concavity of the profit function which resembles to a newsvendor’s profit. They achieve so through establishing a connection between the general failure rate and price elasticity. However, some of their conditions are difficult to verify.

Based on the unimodality that we establish on the profit function with respect to price and its derived function with respect to order quantity, we will analyze three different models for the profit function. In this paper, we provide a solution for the wholesale price only contract; that is, we identify conditions under which both the wholesale price and retail price can be optimally, uniquely, determined. This appears the first attempt to resolve an open problem in the literature.

The second application (in Section 4) considers a joint pricing, sales effort and ordering decision newsvendor-type problem. It is normally assumed that the demand is dependent only on either the price or the sales effort in the literature (e.g., Taylor, 2002; Krishnan et al., 2004), while in our model it is sensitive to both the price and the sales effort. In the current literature the sales-effort cost is assumed either to be independent of the order quantity (see Taylor, 2002; Shum and Simchi-Levi, 2007) or to be dependent on demand (Krishnan et al., 2004), whereas in our setting, the sales-effort cost function is order-quantity dependent. For a comprehensive survey of the sales-effort literature, see Shum and Simchi-Levi (2007). In this application, we introduce a joint pricing and sales-effort problem with a sales-effort cost depending on the order quantity. To the best of our knowledge, this represents a new model in the operations management literature (the motivation of such cost relationship will be articulated later).

We consider a multi-period joint pricing and inventory control problem in the third application (in Section 5). Huh and Janakiraman (2008) derive two unified conditions entirely based on a single-period newsvendor-type profit function that can guarantee the optimality of (s, S)-type policies for a stationary periodic review system with fixed ordering costs. Based on the concavity of the single-period price-setting newsvendor objective function, Song et al. (2009) consider the multiplicative demand function with lost sales, a situation not covered by Huh and Janakiraman (2008). However, by an alternative way we show that under less restrictive conditions (than those of Song et al., 2009) the unimodality of the single-period price-setting profit function can guarantee the two unified conditions raised by Huh and Janakiraman (2008).

We further extend our basic model with the multiplicative demand function in Section 2 to a more general demand function (in Section 6). By introducing a set of new conditions, the unimodality of the objective function can still be retained.

To summarize, our main contributions are two-fold: (1) A general solution framework is presented for the price-setting newsvendor problem with a multiplicative stochastic demand under rather general conditions on the demand function. (2) Based on that general solution framework, we derive new results for three applications.

Throughout the paper, all technical proofs are presented in the Appendix.

2. Unimodality of a newsvendor’s profit function

In this section, we first show the unimodality of a newsvendor’s objective function in the price for any given order quantity and then show the unimodality of the derived profit function in the order quantity.

Specifically, we consider the randomness of demand to be of the multiplicative form, or mathematically, \( D(p) = d(p)x \), where \( d(p) \) is a strictly decreasing function, and \( x \) is a random factor (Petruzzi and Dada, 1999). The newsvendor’s objective is to maximize his expected profit by jointly deciding the order quantity \( y \) and selling price \( p \), formulated as follows:

\[
\max_{p > 0, y > 0} \pi(p, y) = pE[min(y, D(p))] + (\gamma + E[y - D(p)])^+ - wy,
\]

where the first term in the objective function is the expected revenue from sales; the second is the revenue from the leftover
items salvaged at a unit price \( v \geq 0 \) (note \( x^* = \max(x,0) \)); and the third is the total ordering cost with a unit purchasing price \( w \) \((w > v)\) set by his upper-stream supplier.

The probability mass and density functions of the random factor \( z \) in the demand function are \( \Phi(z) \) and \( \phi(z) \), respectively. The support of \( z \) is \((x,\beta)\) with \( 0 \leq x \leq \beta \leq +\infty \). Without loss of generality, we assume \( \Psi(z) = 1 \), which implies \( d(p) = \Psi(dp) \). For ease of exposition, we assume that \( \phi(0) = 0 \) for \( z \in [0,\beta] \cup [\beta,\infty) \). We also assume that \( \lim_{z \to 0^+} \phi(z) = 0 \) if \( x = 0 \). Let \( f(x) = \phi(\beta)/\beta \) denote the failure rate of \( \phi(z) \), and \( g(x) = \phi(x)/\beta \) denote the corresponding generalized failure rate (Lariviere and Porteus, 2001; Lariviere, 2006; Chen, 2011), where \( \overline{\phi}(x) = 1 - \phi(x) \). Note that the latter roughly measures the percentage decrease in the probability of a stockout when the stock quantity increases by one percent.

Before proceeding, we give our two assumptions on the demand function, which are the two central conditions to ensure the unimodality of the objective function.

**Assumption 1 (A1).** The random factor \( z \) has an increasing generalized failure rate (IGFR) in \((x,\beta)\).

The IGFR assumption is satisfied by many commonly used distributions, such as the Uniform, Normal, Exponential, Gamma, and Weibull (with certain restrictions on the shape parameter).

Let \( p = \min(p|d(p) = 0) \). If \( d(p) > 0 \) for all \( p > 0 \), define \( p = +\infty \). Let \( \eta(p) = -d_1/\partial d(p) \) denote the price elasticity of the mean demand \( d(p) \).

**Assumption 2 (A2).** The mean demand \( d(p) \) has an increasing price elasticity (IPE) in \((0,p)\) and satisfies \((1 - V(p)\eta(p)) > 1 \).

IPE is satisfied by commonly seen demand–price relationships, such as \( d(p) = ap - bp \), \( d(p) = ap^b \), and all functions with concave \( d(p) \) and decreasing \( dp/d(p) \). For more discussions, see Yao et al. (2006) and the references therein. To avoid some trivial cases, we also assume the boundary condition \((1 - V(p)\eta(p)) > 1 \). If it is not satisfied, as will be seen in the following Theorem 1, the optimal decision for the news-vendor is to set the price to its upper bound for any order quantity. (For this reason, it is standard to assume \( \eta(p) \geq 1 \) when demand functions take the multiplicative form.)

**Lemma 1.** For \( x \in (x,\beta) \), define \( V(x) = \int_0^x \overline{\phi}(z) dz. \) Then, (i) \( V(x)g(x) > 1 \); (ii) \( V(x) < 0 \); and (iii) \( \lim_{z \to -\infty} V(x) = +\infty \) and \( \lim_{z \to +\infty} V(x) = x \).

It is implicitly assumed in A2 that there exists a minimum \( p^0 < p \) such that \( \eta(p) \geq 1 \) for \( p \in (p^0,p) \). The following lemma establishes the connections between the IPE assumption and those in the literature.

**Lemma 2.** Given \( \eta(p) \geq 1 \), the concavity of \( p\eta(p) \) implies the IPE of \( d(p) \), and IPE in turn implies the convexity of \((1/p)dp \).

Lemma 2 shows that the concavity of \( p\eta(p) \) is more restrictive, while the convexity of \((1/p)dp \) is more general than the IPE, both of which are widely adopted in the literature (e.g., Kocabiyikoglu and Popescu, 2011; Rajan et al., 1992, respectively). Based on the concavity of riskless unconstrained revenue (i.e., \( p\eta(p) \) in the current setting) and certain restrictions on the lost-sales elasticity, Kocabiyikoglu and Popescu (2011) show the joint concavity of news-vendor’s profit function in price and quantity, ensuring the uniqueness of optimal price–quantity decision. However, our analysis will base on the assumptions of IPE and IGFR (see Theorems 1 and 2 below).

Recall that \( \phi(z) = 0 \) for \( z \in [0,\beta] \cup [\beta,\infty) \). For simplicity of exposition, hereafter we further assume \( x = 0 \) and \( \beta = +\infty \), as it is easy to extend to cases with positive \( x \) or finite \( \beta \).

**Theorem 1.** With assumptions A1 and A2, \( \pi(p,y) \) is quasi-concave in \( p \) for any given \( y > 0 \), and there exists a unique optimal \( \hat{p}(y) > \max(p^0, y) \), satisfying

\[
H(p,y) := V\left(\frac{y}{\partial d(p)}\right) - kp = 0,
\]

where \( \hat{p}(y) \) is an optimal price that maximizes the profit with any given \( y \), and \( l(p) = (1 - \frac{1}{p\eta(p)}) \). Moreover, \( \hat{p}(y) \) is strictly decreasing in \( y > 0 \). Let \( Z(y) = y/\partial \hat{p}(y) \), the stock factor. Then, \( Z(y) \) is strictly increasing in \( y > 0 \).

**Lemma 3.** Let \( L(x) = V(x) - l(x) \partial \phi(x) \). Then \( L(x) \) is strictly decreasing, and there exists a unique root \( x^0 \in (0, +\infty) \) for equation \( L(x) = 0 \).

**Theorem 2.** Suppose A1 and A2 hold. Then, \( \pi(\hat{p}(y),y) \) is quasi-concave in \( y > 0 \) and there exists a unique optimal order quantity \( y^* \) satisfying

\[
G(y) = \partial Z(y) - \frac{w - v}{\partial \hat{p}(y)} = 0.
\]

Next, we discuss how to find the optimal price and order quantity by searching the optimal stock factor.

**Corollary 1.** Suppose A1 and A2 hold. There exists a unique optimal stock factor \( x^* \) satisfying

\[
V(x^*) - l(x^*) \partial \phi(x^*) = 0.
\]

As a result, the optimal price \( p^* \) and order quantity \( y^* \) are expressed by \( p^* = (w - v\phi(x^*)/(1 - \phi(\overline{\phi}(x^*))) \) and \( y^* = d(p^*)\overline{\phi}(x^*) \), respectively.

3. Application 1: wholesale price only contract

In this section, we consider a Stackelberg game-setting in which a manufacturer is the leader, whereas a retailer is the follower. The former sets her wholesale price \( w \), taking into account its effect on the retailer’s order quantity. The retailer who faces a price-sensitive demand with a multiplicative random term, decides his order quantity \( y \) and retailing price \( p \) simultaneously based on the wholesale price \( w \).

For any given wholesale price, the retailer’s optimal decisions are characterized in our basic model. We are interested in the optimal decision of the manufacturer, whose objective is to maximize her profit by setting an optimal wholesale price \( w \) as follows:

\[
\pi_w(W) = (W - c)\pi(W),
\]

where \( c \) is the unit production cost, and \( y(W) \) is the retailer’s order quantity for the given wholesale price \( w \).

Under rather general conditions, Lariviere and Porteus (2001) show that the manufacturer’s profit function is unimodal for the case of exogenous retail price. Raz and Porteus (2003) consider a similar setting to ours here but with a finite-state discrete demand function; and they show that there may exist multiple maximizers (of \( w \)) for the profit function. In this section, our aim is to study under what conditions the unimodality (quasi-concavity) of profit function can withhold within the price-setting news-vendor framework as described in the last section.

For convenience of exposition, hereafter in this section we assume the salvage value \( v = 0 \) (see Lariviere and Porteus, 2001). Recall that the optimal stock factor \( x \) must satisfy \( V(x) = \eta(p) \) and \( \overline{\phi}(z) = w/p \) simultaneously, according to Theorems 1 and 2. Specifically, we consider the following two scenarios based on the

\footnote{For finite \( x \) or \( \beta \), such \( x^0 \) exists in \((x,\beta)\).}
monotonicity of $\eta(p)$: Case I: $\eta(p) = b > 1$, and Case II: $\eta'(p) > 0$. The following discussion is based on these two cases.

Case I: $\eta(p) = b > 1$.

In this case, $w = p\beta(z^b)$, where $z^b$ satisfies $V(z^b) = b$. Substituting it into the profit function $\pi_m(w)$, we get the equivalent expression

$$\pi_m(p) = (p - c) d(p) K_0,$$

where $c = \beta(T) z^b$ and $K_0 = z^b T(z^b)$. Hence, in this case, solving the original objective function is equivalent to a price-sensitive deterministic profit maximization problem. It can be easily verified that $1/d(p)$ is convex. Therefore, according to Rajan et al. (1992), $\pi_m(p)$ is quasi-concave in $p$, and there exists a unique optimal solution denoted by $p^*$ satisfying the first order condition. As a result, the optimal wholesale price $w^*$ can be expressed by

$$w^* = p^* T(z^b) = \frac{bc}{b - 1}.$$

Case II: $\eta'(p) > 0$.

Denote by $\eta^{-1}(\cdot)$ the inverse of $\eta(\cdot)$. We then have $p(z) = \eta^{-1}(V(z))$, which is clearly decreasing in $z$. As a result, $w = p(z) T(z)$. Substituting it into the objective function $\pi_m(w)$, we get the equivalent expression:

$$\pi_m(p) = (p(z) T(z) - c) d(p) z.$$

(4)

Define $H_0(z) = 1 - c/p(z) T(z)$, $H_1(z) = H_0(z) - 1/V(z)$, and $H_2(z) = H_0(z) - g(z)$. Straightforwardly, $H_i(z)$, $i = 0, 1, 2$, are all strictly decreasing and their relationships are as follows:

$$H_0(z) > H_1(z) > H_2(z),$$

the last inequality is due to $V(z) g(z) > 1$ by Lemma 1(i). Denote by $z_1$ and $z_2$ the roots of $H_1(z) = 0$ and $H_2(z) = 0$, respectively. Then, it follows that $z_2 < z_1$ due to $H_2(z) < H_1(z)$.

Instead of directly studying the unimodality of $\pi_m(w)$ with respect to $w$, we examine the unimodality of $\pi_m(z)$, the uniqueness of maximizer of which guarantees the uniqueness of optimal wholesale price $w^*$. The following pair of conditions is then sufficient to guarantee the unimodality of $\pi_m(z)$.

Assumption 3 (A3). Let $\rho(p) = \eta(p) (\eta(p) - 1) / \eta p^*_y$. (i) $\rho(p)$ is increasing in $0, p$; (ii) $V(z) - 1 / g(z)$ is decreasing in $[z_2, z_1]$.

For linear demand $d(p) = a - bp$ and exponential demand $d(p) = ae^{-bp}$, we have $p(z) = (2b/a)z - 1$ and $\rho(p) = bp - 1$, respectively, both of which are in increasing in $[0, p]$ and satisfy condition (i), (ii) is also implied by conditions such as decreasing $\eta p^*_y$ (expressed in terms of $\rho(p)$) and strictly increasing price elasticity assumption (i.e., $\eta'(p) > 0$). Condition (ii) is slightly restrictive. Nevertheless, it is satisfied by some common distributions as illustrated below.

Pareto: For a Pareto distribution, as $g(z)$ is a constant and $V'(z) < 0$ due to Lemma 1, condition (ii) in A3 is satisfied.

Uniform: If $\varepsilon$ is uniformly distributed in $[a, 2 - a]$ ($0 < a < 1$), then

$$V(z) - 1 / g(z) = \frac{4a(1 - a)}{z^2 - a^2} + \frac{2(1 - a)}{z + a - 1}.$$

It can be easily verified that if $z > 1 + \sqrt{2} a$, $[2/(z + a) - 1/2] < 0$, which in turn implies if $a < (\sqrt{2} - 1) z_2$, then the above uniform distribution satisfies condition (ii).

Exponential: If $\phi(z) = e^{-z}$, i.e., $\varepsilon$ follows the exponential distribution, then we have

$$\left(\frac{V(z) - 1}{g(z)}\right) = e^{z - 1} - 2(e^{z - 1})^2 + \frac{1}{2} \left(\frac{2(2z^2 - z^2 - 1)}{2ze^z - z^2 - 1} \right).$$

where the inequality holds due to $e^z > 1 + z + z^2/2$. Since $[2(e^z - z - 1) - 2] = 2(e^z - 1) > 0$, there exists a unique $z_0$ such that $V(z) - 1 / g(z)$ is decreasing in $(0, z_0)$ and $2(e^z - z - 1) = z_0^2$, Therefore, if $z_0 \geq z_1$, condition (ii) is satisfied by the exponential distribution.

For other distributions, we conjecture that condition (ii) is satisfied with some restrictions on distribution parameters.

To summarize, we have the following theorem.

Theorem 3. Suppose A1 and A2 hold. The manufacturer’s optimal contract is as follows:

(i) If $\eta(p) = b > 1$, then $w = bc / (b - 1)$.

(ii) If $\eta'(p) > 0$, then $w = p(z_m) T(z_m)$, where $z_m \in [z_2, z_1]$ and satisfies

$$\gamma(z) := \frac{1}{g(z)} H_0(z) + \rho(p(z)) H_1(z) \left(V(z) - \frac{1}{g(z)}\right) - 1 = 0.$$

Moreover, under A3, $z_m$ is unique, which implies the uniqueness of $w^*$.

According to Theorem 3, we establish the optimal distribution-free wholesale contract for the case with constant price elasticity and distribution-dependent wholesale contract for the case with strictly increasing price elasticity under certain conditions. A3 is a pair of restrictions on the mean demand function and distribution, and it is sufficient to guarantee the quasi-concavity of profit function $\pi_m(z)$. However, other sufficient conditions can be developed by replacing $V(z)$ with $\eta(p)$, or vice versa, since $V(z) = \eta(p)$. For instance, we replace A3 by the following pair of conditions: (i) $\rho(p) := (\eta(p) - 1) / \eta p^*_y$ is increasing; (ii) $V(z) - 1 / g(z)$ is decreasing. Clearly, (i) is slightly more restrictive than (i), whereas (ii) is more general than (ii) as $\eta'(p) > 0$ and $V(z) < 0$. Based on the expression of $\gamma(z)$ under the new pair of conditions, the uniqueness of $z_m$ and hence the uniqueness of $w^*$ can be guaranteed. In other words, there appears a tradeoff between the restrictions on the demand function and on the distribution of random term.

4. Application II: joint price and sales-effort model

This section considers a joint pricing, sales effort and ordering decision problem. Our problem has salient features: (1) demand is sensitive to both price and sales effort, whereas it is normally assumed to be dependent on either price or sales effort in the literature (e.g., Taylor, 2002; Krishnan et al., 2004); (2) the sales-effort cost function is proportional to order quantity, whereas it is assumed either to be completely independent of order quantity (see Taylor, 2002; Shum and Simchi-Levi, 2007) or to be dependent on demand (Krishnan et al., 2004) in the current literature. See Shum and Simchi-Levi (2007) for a comprehensive literature review.

Our problem is partially motivated by the following industrial practice: A fresh product, such as seafood and vegetables, distributor purchases a certain quantity of a fresh product from its production origin and sells it at a distant market. The market demand depends on the sales price and freshness level of the products, which in turn depends on the sales effort (e.g., the package of the products, the refrigerated facility of the container, and so on) of the distributor during the logistics process. For example, she can choose different kinds of packages for keeping the products fresh. Evidently, the more refined the package, the higher the unit sales-effort related cost.

We maintain the terms defined in Section 2, except for those related to the sales effort. Denote by $\rho$ the sales effort and $s(\rho)$ its corresponding unit cost. The multiplicative stochastic demand is defined as $D(p, \rho) = dp f(\rho) e$. The objective function can then be
expressed as follows:

\[
\max_{p > 0, y > d_{0} \in \mathbb{R}} \pi(p, y, \rho) = pE(\min(y, \hat{D}(p, \rho))
\]

where \(\pi(p, y, \rho)\) is exactly the newsvendor profit function defined in Section 1.

Define \(\psi(p, y, \xi) = \max(y - d(p)\xi, 0)\) for any realization \(\xi\) of random term \(e\). Huh and Janakiraman (2008) establish a unified condition on the single-period profit function to guarantee the optimality of (\(s, s\)) policy, which under our context can be expressed as follows:

(a) \(\pi(\hat{P}(y, z), y)\) is quasi-concave.
(b) For any \(y^* \leq y^1 \leq y^2\) and \(p^2\), there exists \(p^1\) such that \(\pi(p^1, y^1) \geq \pi(p^2, y^2)\) and \(\psi(p^1, y^1) \leq \psi(p^2, y^2)\) for any \(e\).

Next, we will prove that the price-setting newsvendor with a multiplicative demand satisfies the above conditions (a) and (b). Define \(\hat{P}(y, z) = d^{-1}(y/z)\), where \(z\) is the stock factor defined in the previous section. Clearly, \(\hat{P}(y, z)\) is increasing in \(z\) while decreasing in \(y\).

**Lemma 5.** Let \(\hat{z}_0 = y_0/d(\hat{P}(y_0))\). Then, \(\hat{P}(y, z_0) < \hat{P}(y)\) for \(y > y_0\) and \(\hat{P}(y, z_0) > \hat{P}(y)\) for \(y < y_0\).

Let \(\hat{P}(y, z) = \pi(\hat{P}(y, z), y)\). We introduce the following assumption, which is a key condition for the next lemma.

**Assumption 5 (A5).** \(p + d(p)/d_p\) is increasing in \(p \in (v, p]\).

A5 is equivalent to the curvature of \(d(p)\), defined as \(E(p) = d(p)/d_p^2\), being less than 2 (see Song et al., 2009 and references therein).

**Lemma 6.** Suppose A1, A2, and A5 hold. Let \(y^0/d(p^0) = z_0^0 = \hat{y}(y^0)\) and \(z^0 = \hat{y}(y^0)\). If \(y^0 > y^1 \geq y^0^*, z^0 < z^0^*, \) and \(p^0 > v\), then \(\Pi(y^0, z^0) \geq \Pi(y^0, z^0^*)\) for any \(\hat{y}(y^0, z^0')\).

Noting that \(p + d(p)/d_p = p(1 - 1/\eta(p))\) and recalling that \(\eta(p) > 1\) if \(p < p^0\), A3 is not required in the proof of Lemma 6 in case of \(p^0 \leq v\). For specific demand functions, such as \(d(p) = a - bp\) and \(d(p) = ap^2\), if their corresponding parameters satisfy \(a \geq b > 1\), then A3 is redundant in the ensuing Theorem 5.

**Theorem 5.** Suppose A1, A2, and A5 hold. For any two order quantities \(y^1\) and \(y^2\) such that \(y^* \leq y^1 \leq y^2\) and given selling price \(p^2\), there exists a selling price \(p^1\) such that \(\pi(p^1, y^1) \geq \pi(p^2, y^1)\) and \(\psi(p^1, y^1) \leq \psi(p^2, y^2)\) for any realization \(\xi\), of random term \(e\).

Theorem 5, together with Theorem 2, shows that under Assumptions A1–A2 and A5, Condition 1 of Huh and Janakiraman (2008) is satisfied, which leads to the optimality of (\(s, s\)) policies in an infinite-horizon stationary joint pricing-inventory control system with lost sales and fixed costs.

6. An extension: general demand setting

In this section, we study the unimodality of the price-setting newsvendor problem under a general demand setting. Specifically, we assume \(D(p) = dp(p, e)\). Similar to the case with multiplicative demand, we study the unimodality of \(\pi(p, y, \rho)\) for any given \(y\) and the unimodality of its derived function \(\pi(\hat{P}(y), y)\), where \(\hat{P}(y)\) is the optimal price for a given order quantity \(y\). The following assumption is required to guarantee the unimodality we expect.

**Assumption 6 (A6).** The demand function \(d(p, e)\) satisfies the following: (i) \(d(p, e)\) is strictly concave for any given \(e\); (ii) \(d_p(p, e) < 0\) and \(d_{ee}(p, e) > 0\); and (iii) \(d(p, e) + pd(p, e)\) is
decreasing in ε, where f(ε) is, as defined earlier, the failure rate function of the random term ε.

Assumptions 6(i) and (ii) are also adopted by Kocabiyikoglu and Popescu (2011) when considering a price-setting newsvendor problem under a general demand function. By imposing certain restrictions on the elasticity of lost-sales rate, they show the joint concavity of the newsvendor’s profit function regarding both the price and order quantity, which in turn guarantees the uniqueness of optimal decision. We here, however, give restrictions on the demand function and failure rate directly in Assumption 6(iii), which is neither more restrictive nor more general than those considered by Kocabiyikoglu and Popescu (2011). When f(ε) is increasing (refer to Yao et al., 2006 for more discussions), and d(p, ε) is submodular in p and ε and concave in ε, Assumption 6(iii) is comfortably satisfied. As a result, given that the failure rate is increasing, for the additive-multiplicative demand d(p, ε) = A(p) + B(p)ε, if A’(p) ≤ 0, B’(p) ≤ 0, (pA(ε))’ < 0 and (pB(ε))’ < 0, then the conditions in Assumption 6 are satisfied. Note that (pA(ε))’ and (pB(ε))’ are expected revenues in part, and hence (pA(ε))’ < 0 and (pB(ε))’ < 0 simply mean those revenues being concave, which is not restrictive at all. Moreover, if A(p) = 0 or B(p) = 1, the additive-multiplicative demand form reduces to the multiplicative or additive demand case.

For any pair (p, y), denote by z(p, y) the unique solution to d(p, z) = y. Then, z2 = d0/d1 > 0, and z2 = 1/d1 > 0. Without loss of generality, we assume y = 0 in this section. For ease of expression, we retain the other notations used in Section 2. The objective function can thus be expressed as

\[
\max \pi(y) = pE\{\min(y, d(p, y))\} - wy
\]

\[
= (p-w)y - p \int_0^{z(p, y)} (y-d(p, v)) \phi(v) dv.
\]

Taking the first derivative with respect to p, we have

\[
\ell(p, y) := \frac{\partial \pi}{\partial p} = y - \int_0^{z(p, y)} (y-d(p, v)) \phi(v) dv
\]

\[
- \int_0^{z(p, y)} [-pd_y(p, v)] \phi(v) dv.
\]

**Lemma 7.** With Assumption A6, \(L_y(p, y) < 0\) and \(L_y(p, y)_{p-p(y)} < 0\).

Regarding p for any given y, which in turn implies the uniqueness of optimal price \(\hat{p}(y)\). Let \(\pi(y) = \pi(p(y), y)\) and \(Z(y) = z(p(y), y)\) (with a bit of abuse in notation). Taking the first derivative of \(\pi(y)\) with respect to y, we obtain

\[
\frac{d\pi}{dy} = \frac{\partial \pi}{\partial y} = - \int_0^{z(p, y)} (y-d(p, v)) \phi(v) dv
\]

\[
- \int_0^{z(p, y)} [-pd_y(p, v)] \phi(v) dv.
\]

Similar to the case with multiplicative demand, as illustrated by Theorem 2 in Section 2, if \(\hat{p}(y)\) is decreasing while \(Z(y)\) is increasing in y, then the unimodality of \(\pi(y)\) can be guaranteed. We summarize the major results in Theorem 6.

**Theorem 6.** Suppose Assumption A6 holds. Under a general demand setting with \(D(p) = d(p, \epsilon)\), we have

(i) \(\pi(p, y)\) is strictly concave in p for any given y > 0, and there exists a unique optimal \(\hat{p}(y)\). Moreover, \(\hat{p}(y)\) is strictly decreasing, while \(Z(y)\) is increasing in y > 0.

(ii) \(\pi(\hat{p}(y), y)\) is quasi-concave in y > 0, and there exists a unique optimal order quantity \(y^*\) satisfying \(f|Z(y)| - w/\pi(y) = 0\).

Theorem 6 shows that under certain restrictions on the demand and random factor, the unimodality of the price-setting newsvendor’s objective function can be guaranteed even for a general demand setting. Assumptions 6(i), (ii) are widely adopted in the literature (see Yao et al., 2006). Assumption 6(iii) is new, which encompass the additive-multiplicative demand as a special case. However, for a specific demand model format like multiplicative, less restrictive conditions such as increasing price elasticity and increasing generalized failure rate are enough to guarantee the unimodality of price-setting newsvendor’s expected profit function, as our previous sections illustrate.

### 7. Concluding remarks

In this paper, we have presented a general solution framework for the price-setting newsvendor problem with multiplicative stochastic demand. It comprises two rather general assumptions on the demand function: (1) increasing price elasticity for the mean demand rate, and (2) increasing generalized failure rate for the random term. The result of the unimodality of the newsvendor’s objective function – the expected profit – is then extended to three problems: the wholesale price only contract, a joint pricing, sale-effort and inventory decision, and a dynamic joint pricing-inventory model with lost-sales problem. Furthermore, we find that even under a general stochastic and price-sensitive demand setting, with relatively restrictive yet fairly reasonable assumptions on the demand function, the unimodality property of the newsvendor’s objective function retains.

Possible future research includes: (1) Further improving the conditions; (2) in our second application on the joint pricing, sales-effort and inventory decision problem, we assume that the sales-effort cost function is proportional to the order quantity. In reality, the relationship between the sales-effort cost and the order quantity may be more general, e.g., the sales-effort cost consists of two parts: the fixed effort cost independent of the order quantity, and the unit variable effort cost proportional to the order quantity. The question that remains is what additional conditions are needed to guarantee the uniqueness of the optimal sales effort under such a general sales-effort cost assumption. This is a problem worthy of further investigation.

**Appendix A**

**Proof of Lemma 1.** Based on Assumption 1, we have

\[
\int_0^x \xi \phi(\xi) d\xi = \int_0^x \phi(\xi) d\xi \leq g(x) \int_0^x \phi(\xi) d\xi,
\]

which implies \(1/g(x) \leq \int_0^x \phi(\xi) d\xi / \int_0^x \xi \phi(\xi) d\xi\). Hence, if \(x \geq 0\), then

\[
V(x) = \frac{\alpha}{\int_0^x \xi \phi(\xi) d\xi} + \frac{\int_0^x \phi(\xi) d\xi}{\int_0^x \xi \phi(\xi) d\xi} \geq \frac{\alpha}{\int_0^x \phi(\xi) d\xi} + \frac{1}{g(x)} = \frac{1}{g(x)}.
\]

If \(x = 0\), since \(\lim_{x \to 0} \phi(x) = 0\), we have \(g(0) = 0\), which implies "<" in (7) should be "<". These two scenarios establish our conclusion (i). As a result,

\[
V(x) = \frac{V(0) - 1}{x} [1 - V(0)g(x)] < 0,
\]

which establishes (ii).
If \( x > 0 \), based on the expression \( V(x) = \alpha / \int_0^x \xi \phi(\xi) \, d\xi + \int_0^x \Phi(\xi) \, d\xi / \int_0^x \xi \phi(\xi) \, d\xi \), it is straightforward that \( \lim_{x \to \infty} V(x) = +\infty \). Otherwise, we have:

\[
\lim_{x \to -\infty} V(x) = \lim_{x \to -\infty} \frac{\Phi(\xi)}{\int_0^x \xi \phi(\xi) \, d\xi} = \frac{1}{\theta(0^-)} = +\infty,
\]

the last equality of which is due to our assumption \( \lim_{x \to 0^-} \phi(x) > 0 \).

Since it is assumed that \( \theta(0) = 1 \), we have:

\[
\lim_{x \to -\infty} V(x) = +\alpha + \int_0^\theta \Phi(\xi) \, d\xi = \int_0^\theta \xi \phi(\xi) \, d\xi = 1. \quad \square
\]

**Proof of Lemma 2.** Taking derivative of \( p(d|p) \) regarding \( p \), we have \( p(d|p)/p = \partial p + d/p = d/p[1 - \eta(p)]. \) Consequently, we have:

\[
\partial p = \partial p[1 - \eta(p)] \geq 0,
\]

the last inequality of which is due to \( \eta(p) \geq 1 \) and \( d_\phi < 0 \).

In contrast, taking the first derivative of \( 1/d(p) \) regarding \( p \), we have \( (1/d(p))' = -d_p/p^2 = \eta(p)/p(d/p). \) Thus, based on conditions \( \eta(p) \geq 0 \) and \( \eta(p) \geq 1 \), we have:

\[
\left( \frac{1}{d(p)} \right)' = \frac{1}{p(d(p))} \eta(p) + (\eta(p) - 1) \eta(p) \geq 0,
\]

which implies the convexity of \( 1/(d(p)) \) and this completes our proof. \( \square \)

**Proof of Theorem 1.** The objective function can be transformed into:

\[
\pi(p,y) = (p - \xi)y - (p - v) \int_0^{y - \partial p} (y - d(p)\xi) \phi(\xi) \, d\xi.
\]

For any fixed \( y \), taking partial derivative with respect to \( p \) on both sides of (8), we have:

\[
\frac{\partial \pi(p,y)}{\partial p} = y - \xi \phi\left( \frac{y}{d(p)} \right) + d(p) \int_0^{y - \partial p} \phi(\xi) \, d\xi + (y - v)d_p \int_0^{y - \partial p} \phi(\xi) \, d\xi = d(p) \int_0^{y - \partial p} \phi(\xi) \, d\xi \left[ \frac{y}{d(p)} - 1 - \frac{v}{p} \right] \eta(p),
\]

\[
= d(p) \int_0^{y - \partial p} \phi(\xi) \, d\xi \frac{y}{d(p)} H_p(p,y). \]

Note that \( \partial \pi(p,y)/p \) has the same sign as \( H_p(p,y) \). If \( p \leq p_0 \) or \( p \leq v \), then \( H_p(p,y) < 0 \), indicating that an increase in \( p \) is beneficial. We also have \( H_p(p,y) < 0 \) for \( p \in (v,\infty) \) and \( H_p(p,y) < 0 \) due to Lemma 1. For any given \( y \), based on Assumption A2, \( H_p(p,y) = 1 - \eta(p) \), which together with \( H_p(p,y) < 0 \) for \( p < p_0 \) or \( p < v \) implies that there exists a unique optimal \( \hat{P}(y) \) satisfying \( \hat{P}(y) > \max(p_0,v) \) and \( H(\hat{P}(y),y) < 0 \), or equivalently, Eq. (1) holds. Furthermore, by the Implicit Theorem, we have:

\[
H_p \frac{d\hat{P}(y)}{dy} + H_p \frac{d\hat{P}(y)}{dy} = \frac{d\hat{P}(y)}{dy} = -\frac{H_p}{H_p} < 0.
\]

Recall that \( \hat{P}(y) > v \) and that it is decreasing in \( y \). Thus, we have:

\[
y \mapsto \hat{P}(y) \mapsto \left( 1 - \frac{v}{\hat{P}(y)} \right) \eta(\hat{P}(y)) \mapsto \left( y/d(\hat{P}(y)) \right) \mapsto Z(y) \mapsto Z(y),
\]

the last "\( \mapsto \)" is due to the fact that \( V(\cdot) \) is decreasing. \( \square \)

**Proof of Lemma 3.** As \( (w - v)\Phi(x)/\Phi(x) = (w - v)/\Phi(x) \cdot v > v \) and it is increasing in \( x \), \( \lim_{x \to \infty} \Phi(x)/\Phi(x) \) is increasing in \( x \). This, together with the strictly decreasing property of \( V(\cdot) \), implies that \( L(x) \) is strictly decreasing in \( x \). According to Lemmas 1 and 2, we have:

\[
\lim_{x \to \infty} V(x) = V(0) - \lim_{x \to \infty} V(x) = +\infty > 0,
\]

and:

\[
\lim_{x \to \infty} V(x) = V(0) - \lim_{x \to \infty} V(x) = +\infty > 0,
\]

which implies the existence of a unique \( x^* \in (0, \infty) \) such that \( L(x) = 0 \). \( \square \)

**Proof of Theorem 2.** Taking the first derivative with respect to \( y \) on the profit function \( \pi(p,y) \), we have:

\[
\frac{d\pi(p,y)}{dy} = \frac{\partial \pi}{\partial p} \frac{dp}{dy} + \frac{\partial \pi}{\partial y} = \left[ (p-v) - (p-v) \Phi\left( \frac{y}{d(p)} \right) \right] = (\hat{P}(y) - v) G(y).
\]

Note that if \( \hat{P}(y) > v \), \( d\pi(p,y)/dy \) has the same sign as \( G(y) \).

Based on Theorem 1, it is straightforward that \( G(y) \) is decreasing in \( y \). Therefore, \( \pi(p,y) \) is quasi-concave and unimodal in \( y > 0 \) and there exists a unique optimal \( y^* \) satisfying either the first order condition (2) or the boundary conditions (i.e., \( y^* = 0 \) or \( y^* = +\infty \)).

Letting \( p_0 = (w - v)\Phi(x)/\Phi(x) \) and \( y^* = (p_0 - x^0) \Phi(x) \), we have by their definitions \( V(y^*) = d(p_0)/p_0 \), which implies \( p_0 = \hat{P}(y^*) \) and \( x^0 = V(y^*) \). As a result, \( G(y^*) = 0 \). Due to the uniqueness of optimality of \( \pi(p,y) \), we know that there exists a unique optimal \( y^* \) that satisfies Eq. (2). \( \square \)

**Proof of Corollary 1.** It follows directly from Lemma 3 and Theorem 2. \( \square \)

**Proof of Theorem 3.** Part (i) has been shown earlier. We thus only consider Part (ii). Taking the first derivative regarding \( z \) on both sides of \( V(z) = \eta(p) \), we have \( V'(z) = \eta(p) dp/dz \), which implies:

\[
\frac{dp}{dz} = \frac{V'(z)}{\eta(p)} = \frac{V(z) - 1 - V(z)}{\eta(p) - 1} - \frac{V(z)}{\eta(p)} = \frac{1}{\eta(p)} \left[ V(z) - 1 - V(z) \right].
\]

Consequently:

\[
\frac{d\pi_m(z)}{dz} = \left[ z \Phi(z) dp(dz) + (p\Phi(z) - c) dz dp + \frac{dp}{dz} \phi(z) dp(dz) + [p\Phi(z) - c] dp \right] \frac{dp}{dz} = \left[ \rho(p(z)) H_1(z)(1 - V(z)) + H_2(z), \right.
\]

where \( \pi_m(z) = (1/g)\pi_m(x) + \rho(p(z))H_1(z)(1 - g(z)) - 1 \). As a result, the sign of \( d\pi_m(z)/dz \) is the same as the function \( \rho(p(z))H_1(z)(1 - g(z)) - 1 \). On the contrary, the optimal \( z^* \) must lie in \([z_2, z_1]\). The sign of \( d\pi_m(z)/dz \) is also the same as the function \( \pi_m(z) \), which based on A3 is strictly decreasing in \( z \). Thus, \( \pi_m(z) \) is quasi-concave in \( z \), and there exists a unique optimal solution \( z_m \) satisfying \( \pi_m(z) = 0 \). \( \square \)

**Proof of Lemma 4.** For any fixed \( p > 0 \), according to Theorems 1 and 2, \( z^*(p) \) and \( p^*(p) \) must satisfy equations \( V(z) = k(p) \), and
\( \Phi(z) = (p - s)(z - w)/(p - v) \) simultaneously. Denote by \( V^{-1}(\cdot) \) the inverse of \( V(\cdot) \). Based on \( V(\cdot) = \lambda(\cdot) \), we have \( z = V^{-1}(\lambda(\cdot)) \).

Substituting it into \( \Phi(z) = (p - s)(z - w)/(p - v) \) and by some algebra transformation, we have

\[
(p - v)\overline{V}^{-1}(\lambda(\cdot)) = s(\cdot) + w - v,
\]

the left hand side of which is increasing with respect to \( p \), while the right hand side of which is increasing with respect to \( \lambda(\cdot) \) due to A5. As a result, it follows that \( dp(\lambda(\cdot))/dp \geq 0 \), which in turn implies \( dz(\cdot)/dp \leq 0 \) because \( z = V^{-1}(\lambda(\cdot)) \).

**Proof of Theorem 4.** Based on Lemma 4 and taking the first derivative with respect to \( \rho \) on \( \Pi(\rho) \), we have

\[
\frac{d\Pi(\rho)}{d\rho} = \left[ -s'(\rho) + (p - v) - d(p)/dp \right] \frac{\Phi(\rho)}{\Phi(\rho)} \frac{\partial \Phi(\rho)}{\partial \rho} \frac{\partial \Phi(\rho)}{\partial \rho} = d(\partial \Phi(\rho)/\partial \rho) \left[ -s'(\rho) + (p - v) - d(p)/dp \right] \frac{\Phi(\rho)}{\Phi(\rho)} \frac{\partial \Phi(\rho)}{\partial \rho} \frac{\partial \Phi(\rho)}{\partial \rho}.
\]

Based on Assumption 4, \( e(\rho(\cdot)) + (w - v)/\rho(\cdot) \) is strictly decreasing in \( \rho \), whereas \( e(\rho(\cdot))\Phi(V^*(\rho)) - 1 \) is strictly increasing due to Assumption 4, Lemmas 1, and 4. As a result, we have \( d^2 \Pi(\rho)/d\rho^2 |_{d\Pi(\rho)/d\rho = 0} < 0 \), which implies \( \Pi(\rho) \) is quasi-concave in \( \rho \), and there exists a unique optimal effort \( \rho^* \) in \( [\rho, \rho]^n \).

**Proof of Theorem 5.** For any \( y > y_0 \) based on the increasing property of \( Z(y) \) by Theorem 1, we have

\[
\frac{y}{d\hat{p}(y)} > \frac{y_0}{d\hat{p}(y_0)} = z_0 = \frac{y}{d\hat{p}(y_0)}.
\]

which thus results in

\[
\Pi(y^*, z^*) - \Pi(y^*, z^*) \geq \Pi(y^*, z^*) - \Pi(y^*, z^*).\]

As \( y > y^* > y^* \) according to Theorem 2, we have

\[
\Pi(y^*, z^*) = \pi(\hat{p}(y^*), y^*) = \pi(\hat{p}(y^*), y^*) = \pi(\hat{p}(y^*), y^*) = \Pi(y^*, z^*).
\]

As a result, \( \Pi(y^*, z^*) - \Pi(y^*, z^*) = 0 \), and this completes our proof.

**Proof of Theorem 5.** Let \( z^2 = (p, y^2)/d(\hat{p}(y)) \) and \( z^2 = (p, y^2)/d(p) \). We consider the following three cases based mainly on the relationship between \( z^2 \) and \( z^1 \).

**Case 1:** \( z^2 \geq z^1 \).

Let \( p^1 = \hat{p}(y^1) \). Then we have \( \psi(p^1, y^1) = y^1[1 - \xi(z^1)]^2 \) for \( i = 1, 2 \), which is increasing with respect to both \( y^1 \) and \( z^1 \). As \( z^2 \geq z^1 \) and \( y^2 > y^1 \), we have \( \psi(p^1, y^2) \leq \psi(p^1, y^1) \). In contrast, \( n(p^1, y^1) = \pi(\hat{p}(y^1), y^1) = \pi(\hat{p}(y^1), y^1) = \pi(p^1, y^1) \) since \( y^2 \geq y^1 < y^2 \).

**Case 2:** \( z^2 < z^1 \) and \( p^2 > p^1 \).

Let \( p^2 = \hat{p}(y^2) \). Then, according to Lemma 6, we have \( \pi(p^2, y^2) = \pi(\hat{p}(y^2), y^2) > \pi(\hat{p}(y^2), y^2) = \pi(p^2, y^2) \). In contrast, we have \( \psi(p^1, y^1) = y^1[1 - \xi(z^1)]^2 < y^1[1 - \xi(z^1)]^2 = \psi(p^2, y^1) \).

**Case 3:** \( z^2 < z^1 \) and \( p^2 < p^1 \).

Let \( p^1 = p^2 \). Then \( \psi(p^1, y^1) = y^1[1 - \xi(z^1)]^2 < y^1[1 - \xi(z^1)]^2 = \psi(p^2, y^1) \).

In contrast, as \( \psi(\hat{p}(y^1), y^1) = (y^1 - d(p)/dp)^2 \xi^2 < (y^1 - d(p)/dp)^2 \xi^2 = \psi(p^2, y^1) \).

Let \( p^1 = p^2 = p^1 \). Then \( \psi(p^1, y^1) = y^1[1 - \xi(z^1)]^2 < y^1[1 - \xi(z^1)]^2 = \psi(p^2, y^1) \).

\[ \psi(p^1, y^1) = y^1[1 - \xi(z^1)]^2 < y^1[1 - \xi(z^1)]^2 = \psi(p^2, y^1) \).

where the last inequality is due to A7(i) and A7(ii).

On the other hand, we have

\[
\begin{align*}
\mathcal{L}(p, y) &= y + \int_0^{\text{d}(p, y)} (y - \text{d}(p, y)) d(\mathcal{V}(y)) - \int_0^{\text{d}(p, y)} (p - \text{d}(p, y)) d(\mathcal{E}(y)) \\
&= d(p, 0) - \int_0^{d(p, 0)} d(\mathcal{E}(y)) d(\mathcal{V}(y)) - \int_0^{d(p, 0)} d(\mathcal{E}(y)) d(\mathcal{V}(y)) = d(p, 0) - \int_0^{d(p, 0)} d(\mathcal{E}(y)) d(\mathcal{V}(y)) = d(p, 0) - \int_0^{d(p, 0)} d(\mathcal{E}(y)) d(\mathcal{V}(y)) = 0,
\end{align*}
\]

which completes our proof.

**Proof of Theorem 6.** Since \( \mathcal{L}(p, y) < 0 \) based on Lemma 7, \( \mathcal{L}(p, y) \) is strictly concave in \( p \) for given \( y \) and there exists a unique optimal price \( \hat{p}(y) \), that satisfies \( \mathcal{L}(\hat{p}(y), y) = 0 \). By Implicit Theorem, we have

\[
\mathcal{L}(p, y)|_{p = \hat{p}(y)} = \hat{p}(y) + \mathcal{L}(p, y)|_{p = \hat{p}(y)} = 0,
\]
which in turn implies that \( \dot{P}(y) = -[\mathcal{L}(p,y)/\mathcal{L}_p(p,y)] \leq 0 \), according to Lemma 7.

\[ \mathcal{L}(\dot{p})(y) = 0 \] is equivalent to

\[ y\mathcal{T}(Z(y)) + \int_0^{Z(y)} (d(p,v) + pd_p(p,v)\phi(v)) \, dv \bigg|_{p = \ddot{p}(y)} = 0. \tag{10} \]

Taking the first derivative regarding \( y \) on both sides of Eq. (10), we have

\[ \mathcal{T}(Z(y)) + [pd_p(p,Z(y))\phi(Z(y))Z'(y)] + \int_0^{Z(y)} (2dp(p,v) + pd_{pp}(p,v)\ddot{p}(y)\phi(v)) \, dv |_{p = \ddot{p}(y)} = 0, \]

which is equivalent to

\[ Z'(y) = \left[ \frac{\mathcal{T}(Z(y)) + \int_0^{Z(y)} (2dp(p,v) + pd_{pp}(p,v)\ddot{p}(y)\phi(v)) \, dv}{pd_p(p,Z(y))\phi(Z(y))} \right] |_{p = \ddot{p}(y)} > 0, \]

where \( (2dp(p,v) + pd_{pp}(p,v)) < 0 \) due to assumption A6(i) and \( dp_p(p,Z(y)) < 0 \) due to assumption A6(ii). Based on Eq. (6), it is straightforward that the monotonicity in \( \ddot{p}(y) \) and \( Z(y) \) can guarantee the unimodality of \( \pi(\ddot{p}(y), y) \) regarding \( y \). This completes our proof. \( \Box \)

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