Comparison of two periodic review models for stochastic and price-sensitive demand environment

Saibal Ray a, Yuyue Song b,*, Manish Verma b

a Desautels Faculty of Management, McGill University, QC, Canada H3A 1G5
b Faculty of Business Administration, Memorial University of Newfoundland, NL, Canada A1B 3X5

ABSTRACT

In this paper, we study two periodic review inventory models which primarily differ in terms of how backordering cost is charged: time-independent backordering (TIB) model where the backordering cost is charged per unit backordered and is independent of the length of time for which backorders persist; and the time-dependent backordering (TDB) model where the backordering cost is charged based on the number of backorders as well as the length of time for which they are on the books (i.e., it is charged per unit per unit time). Our objective is to investigate the impact of these two different backordering cost structures on the optimal decisions of a firm in a stochastic and price-sensitive demand environment. In order to do so, we first develop a general framework, where both such costs exist, in a profit maximizing context. Subsequently, we analyze two special cases of this general framework with either one of the costs—that is, TIB and TDB models—and derive some analytical results regarding the values and behavior of the optimal decisions for both of them. We then concentrate on comparing the two models through extensive numerical experiments. Our investigation demonstrates that the TIB model generally results in longer review periods and lower retail prices. As far as the base stock level is concerned, we show that it can be higher in either setting; however, the safety stock is considerably lower for the TIB model. Lastly, indeed if a firm’s backordering cost is indeed time-dependent, then use of the TIB model for making decisions results in significant profit penalty under most market/operating conditions (specifically for innovative products), except when demand uncertainty and/or the backordering cost are quite low (i.e., for mature, commodity products).

1. Introduction

The continuing rise of globalization and rapid technological advancements has made effective supply chain management (SCM) a core competency in modern business environment. The two most critical and strategic SCM decisions are setting prices and managing inventory (Stern and El-Ansary, 1992). There are two techniques discussed in the literature regarding inventory management for random demand—continuous review and periodic review (see Silver et al., 1998). In the former case, the inventory level is reviewed continuously and a new order is placed whenever this level reaches the reorder point. Normally, all orders are of the same size, but the time interval between orders is random. On the other hand, for periodic review systems, the inventory level is reviewed only at regular intervals of time, and at that point orders are placed to bring the stock position up to the base-stock level1; so, order sizes for different periods vary, but the time between orders (i.e., review length) remains constant. In reality, suppliers tend to prefer regularity in terms of order intervals. So, periodic review systems are quite popular among managers (Silver et al., 1998). In this paper we focus on a periodic review system for inventory management. Moreover, in the tradition of recent integrated operations-marketing literature (Yano and Gilbert, 2003), we also take into account the interactions between pricing and inventory decisions while deciding on the optimal strategy for a firm.

There is a significant literature related to different aspects of periodic review systems like random yield (Wang and Gerchak, 1996), emergency replenishments (Bylka, 2005) and variable purchasing costs (Gavirneni, 2004). The inventory modeling framework in recent dynamic pricing and inventory control literature is also a periodic review one (e.g., refer to Polatoglu and Sahin, 2000; Chen and Simchi-Levi, 2004; Huh and Janakiraman, 2008; Song et al., 2009, and references therein).

* Corresponding author.
E-mail addresses: saibal.ray@mcgill.ca (S. Ray), ysong@mun.ca (Y.R. Song), mverma@mun.ca (M. Verma).

1 Assuming the fixed cost for ordering to be zero.

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However, the primary decision variables in all these papers are inventory control and/or retail price. The review cycle length is assumed to be exogenously given. Nevertheless, one of the most fundamental operational issues in periodic review paradigm is: What should be the right review frequency for a particular system? In this paper we consider a periodic review inventory system for a single item where the demand process is stationary, stochastic, and price sensitive. In our setting, the firm needs to simultaneously determine the profit maximizing review cycle length, base stock level and retail pricing strategies.

The key feature of our model framework is the backordering cost structure. In a large proportion of real-life supply contracts, backordering penalties are assessed based on the length of the delay. The usual contract specifications related to delivery delays are as follows:

...If the supplier defaults on his delivery commitments, he shall pay a contractual penalty amounting to—% of the value of the relevant order per day (or week) of the delay to the orderer.

Such penalty clauses appear in a variety of industrial sectors like automotive components, large machineries, oil and gas, aluminium, electronic equipment and retail as well as in government contracts. As has been indicated in academic literature, time-dependent penalties are also the norm for steel mills (Gupta and Wang, 2007) and for maintenance/repair items (Silver et al., 1998). In fact, recent theoretical inventory management assumes that customers care more about how long they have to wait compared to whether or not they have to wait, i.e., based on backordering cost per unit per unit time (Zipkin, 2000). Nevertheless, there are situations in real-life where backordering cost depends only on the number of backorders and not on their lengths (refer to Silver et al., 1998 for examples). Motivated by above, we analyze and compare two backordering cost scenarios in this paper—one where backordering cost is calculated based only on the number of orders backordered in each period and ignores the time duration for which backorders persist (time-independent backordering (TIB model)) and the other where backordering cost is calculated based on how long each backorder remains on the book in each period (time-dependent backordering (TDB) model). One of the primary questions that we investigate in this context is that, if indeed the backordering cost is time-dependent but the TIB model is used for decision-making, how does it affect managerial decisions and what is the extent of profit penalty that the firm will incur.

Although not very popular, there has been some research in inventory management literature investigating the concurrent determination of the optimal review cycle length and the optimal base stock level.

### 2. Model framework

Demand function: Our model framework is based on a retailer selling a single item to end customers in a risk-neutral setting.
The retail demand function per unit time $\xi(1)$ comprises two elements. One part is $D(p)$, a deterministic function of the retail price $p$. Note that $D(p)$ is non-negative and strictly decreasing on $(0, P^*)$, where $P^*$ is the maximal retail price such that $D(p)$ is positive on $(0, P^*)$. The second part is $\epsilon$, a non-price-sensitive, positive random variable having support on $(0,\infty)$, with mean $\mu$ and standard deviation $\sigma$. We specifically assume that $\xi(1) = D(p)\epsilon$. Such multiplicative demand form is quite common in the literature and is supported by empirical study (refer to Petruzzi and Dada, 1999; Granot and Yin, 2005). Without loss of generality, we assume $\mu = 1$ in this paper. Hence the mean and standard deviation of the per unit time demand random variable $\xi(1)$ are $D(p)$ and $D(p)\sigma$, respectively. In addition, suppose that $D(p)$ exhibits the following properties:

**Assumption 1.** $D(p)$ is continuously differentiable and strictly decreasing on $(0, P^*)$. Furthermore, $D(p)/D'(p) = k_1 + k_2p$ where $k_1 \leq 0$ and $k_2 > -1$ are two constants.

Note that the above assumption is not at all restrictive. It means that the curvature of $D(p)$, defined as $E = D(p)D''(p)/D'(p)^2$, is equal to $1 - k_2p\epsilon$, i.e., the expected demand function has a constant curvature and it is not too high ($E < 2$). Most of the demand functions used for studying joint pricing-inventory decisions, e.g., iso-elastic, exponential and linear, do exhibit constant curvatures (refer to Petruzzi and Dada, 1999; Granot and Yin, 2005; Song et al., 2008).

**Inventory management:** The inventory status is reviewed at some equally distributed discrete time points over the planning horizon and a fixed cost of $K(>0)$ is charged for each review. The time constant duration between any two adjacent review points denoted by $t$ is a decision variable. Since there is no fixed cost associated with ordering, a base-stock policy is used for inventory control. That is, after each review, a replenishment order is placed, if necessary, to raise the inventory position up to the base stock level $S$, and the order is received after a lead time $L$, a positive constant. The per unit procurement cost is $c(>0)$. The inventory holding cost is charged for any on-hand inventory at the rate of $h$ per unit time per unit item. On the other hand, all excess demands are backordered, and a backordering cost is charged. The primary difference between the two models is in the way this back-ordering cost is charged; we will discuss this later on. Note that in the following analysis, for analytical convenience, we use the time period between arrivals of two successive orders, rather than between placements of two successive orders, for calculating inventory costs (since $L$ is constant, the two formulations are equivalent).

The order-up-to base stock level $S$ consists of two components: the average demand over the protection interval $t + L$, and the safety stock to reduce the risk of stock out. Suppose that the random variable representing demand during $t + L$ is denoted by $\xi(t + L)$. Clearly, this random variable has a mean of $(t + L)D(p)$ and a standard deviation of $\sigma\sqrt{t + L}D(p)$. So, $S$ can be expressed as

$$S = \{(t + L)/2 + 2\sigma\sqrt{t + L}D(p),$$

where $z$ represents the safety stock factor. In this paper, we approximate $\xi(t + L)$ by a normal distribution $N \sim \xi(t + L) / D(p), r\sqrt{t + L}D(p)$. For tractability reason, a common approach in the inventory literature is to use such a second-order approximation for the demand during the protection interval, i.e., it is assumed to be normally distributed with the precise mean and variance values given by $(t + L)D(p)$ and $\sigma\sqrt{t + L}D(p)$, respectively (Hadley and Whitin, 1963; Silver et al., 1998; Zipkin, 2000, Chapter 7; Axxaster, 2000, Chapters 3 and 5). Tyworth and O'Neill (1997) have shown this approximation to be quite robust (also refer to Lau and Lau, 2003) for a detailed discussion about the accuracy of the approximation). Our research adds to the complexity, compared to the inventory models, in the form of pricing. Moreover, since our goal is to develop managerial insights, rather than accurate decision support systems, the above approach is reasonable for this paper. We denote the density and cumulative functions of the standard normal distribution by $f(z)$ and $F(z)$, respectively. Clearly, for a given $t, L$ and $D(p)$, there is an one-to-one relation between $S$ and $z$. So, the relevant decision variables for us are the length of the review cycle, $t$, the safety stock factor, $z$, and the retail price of the item, $p$. The objective is to maximize the average profit (i.e., per unit time) for this single item system by choosing the optimal values of these three decisions.

Based on the above notations, we are ready to formulate the average total profit function $\pi(t,z,p)$ in terms of $(t, z, p)$. We start by developing a general model framework where the backordering cost depends both on the number of backorders at the end of the period as well as on the length for which backorders persist. Specifically:

$$\pi(t,z,p) = (p - c)D(p) - \frac{1}{r}[K + h(t,z,p) + b_1B_1(t,z,p) + b_2B_2(t,z,p)],$$

where $l(t,z,p)$ is the average on-hand inventory level over one review cycle of length $t$, $h$ is the holding cost per unit per unit time, $B_1(t,z,p)$ is the average number of backorders at the end of a review cycle, $b_1$ is the backordering cost per unit charged on the average number of backorders at the end of a review cycle, $B_2(t,z,p)$ is the average backordering level over one review cycle of length $t$ and $b_2$ is the backordering cost per unit per unit time charged on the backorders during a cycle ($B_2(t,z,p)$ can be thought of as weighted backordering level where the weight on each backorder is the length of time for which that particular backorder is on the books).^7

As discussed before, in this paper, we focus on analyzing two special cases of the above general framework—time-independent backordering (TIB) model in which case $b_2 = 0$ and time-dependent backordering (TDB) model for which $b_1 = 0$. Our aim is to investigate the effects of different backordering cost structures on the values and behavior of the optimal decisions as well as profits by comparing TIB and TDB models. The TIB model was first proposed by Hadley and Whitin (1963), and is driven by the following two assumptions:

- Backordering cost is charged per unit backordered, independent of the time duration for which the demand is back-ordered.
- Backorders are incurred only in very small quantities. This implies that when a new replenishment order arrives, it is almost always sufficient to meet any outstanding backorders.^8

^5 Economically speaking, $E$ is the elasticity of the slope of inverse demand $p(D)$. Loosely, one might think of the assumption that $E$ is a constant to be a smoothness condition on the demand function. Furthermore, $E < 2$ implies that demand should not be too convex. This assumption ensures that the firm’s marginal revenue is downward-sloping, and hence guarantees an interior solution. Refer to Bresnahan and Reiss (1985) and Ritz (2005) for more details.

^6 Examples include concave functions like $D(p) = (a - kp^2)$ ($a > 0, k > 0, 0 < \gamma < 1$), as well as convex ones like $D(p) = ap^{-\gamma}$ ($a > 0, k > 1$), $D(p) = (a - kp^2)^\gamma$ ($a > 0, k > 0, \gamma \geq 1$ or $a > 0, k < 0, \gamma < -1$), $D(p) = ae^{-kp}$ ($a > 0, k > 0$).

^7 Although most periodic review models with the length of the review period as a decision variable assume $b_2 = 0$, there are continuous review inventory model papers where backordering cost includes both $b_1$ and $b_2$ (e.g., Josell et al., 2006 and references therein).

^8 This means $l(t,z,p)$ is also approximate in the TIB model; see below for more details.
Note that the TIB model takes the time duration into consideration while calculating the holding cost (i.e., it is charged per unit per unit time). On the other hand, in the TDB model, both backordering and holding costs are charged per unit per unit time. In the next two sections we present the average profit expressions for the two models.

2.1. Time-independent backordering (TIB) model

In this model: (i) the backordering cost depends only on the average number of backorders at the end of a review period, i.e., the average number of backorders at the end of a review cycle is determined and each of them is charged a cost of \( b \) and (ii) the holding cost is approximated based on the assumption that the number of backorders is small.\(^9\) Based on the expressions for the average inventory holding and backordering costs in Hadley and Whitin (1963), the profit per unit time for this model can be expressed as

\[
\Gamma(t,z,p) = pD(p) - \left( \frac{K}{t} + \frac{tD(p)}{2} \right) \left( h + hz\sqrt{t} + LD(p) \right) + \frac{b}{t} \int_{-\infty}^{z} \left[ S - s_{f}(x) \right] dx + cD(p),
\]

where \( f(x) \) is the density function of \( \xi(t+L) \). In the above equation the first term is for the revenue, the second term represents the review cost, the third and fourth terms are for cycle and safety stock holding costs, respectively, the fifth one represents the backordering cost, while the last one denotes the procurement cost. Based on normal approximation demand during \( (t+L) \), i.e., \( \xi(t+L) \sim N(t,L)D(p),\sigma\sqrt{t+L}D(p) \), the above expression can be rewritten as

\[
\Gamma(t,z,p) = (p-c)D(p) - \left( \frac{K}{t} + \frac{tD(p)}{2} \right) \left( h + hz\sqrt{t} + LD(p) \right) + b\sigma\sqrt{t+L} \left[ f(z) - z \right] D(p) + \frac{b}{t} \int_{0}^{t} \left[ B(t,z,t) - F(t) \right] dt,
\]

where \( f(z) \) and \( F(z) \) are the density and cumulative distribution functions of the standard normal distribution. As indicated before, the firm wants to maximize \( \Gamma \) by optimally selecting the review period \( t \), the safety stock factor \( z \) and the retail price \( p \).

2.2. Time-dependent backordering (TDB) model

In this model: (i) each backorder is charged \( b \) for each unit of time it remains unfulfilled; and (ii) the number of backorders can be arbitrarily large so that a new order might not be able to satisfy all backorders. When the firm places a procurement order at time \( t \), it raises the inventory position to \( S \). In this case the inventory level at \( t+L+\tau \) is \( S - \xi(t+L+\tau) \) for any \( t \in [0,T] \). The total holding and backordering cost rate of net inventory at the time point \( t+L+\tau \) can then be expressed as:

\[
h \int_{-\infty}^{\xi(t+L+\tau)} [S-x] f(x) dx + b \int_{-\infty}^{\xi(t+L+\tau)} [S-x] f(x) dx,
\]

where \( f(x) \) is the density function of the total demand \( \xi(t+L) \) over a time duration of \( \tau \) in the interval \( [0 \leq \tau \leq t] \). For simplicity, let

\[
\delta = \frac{z\sqrt{t+L}}{\sqrt{t+L}} + \frac{1 - \tau}{\sqrt{t+L}} \frac{\sigma}{\delta},
\]

Recall that \( \xi(t+L) \sim N(t+L,\sigma^2) \). This implies that the total holding cost rate of net inventory at \( t+\tau+L \) can be rewritten as

\[
h \int_{-\infty}^{\delta} [S-x] f(x) dx = hD(p) \int_{-\infty}^{\delta} [(t-\tau) + \sigma(\sqrt{t+L} - y) + \sqrt{t+L}]f(y) dy.
\]

Similarly, the total backordering cost rate of the net inventory at \( t+\tau+L \) can be rewritten as

\[
b \int_{-\infty}^{\delta} [S-x] f(x) dx = bD(p) \int_{-\infty}^{\delta} [(t-\tau) + \sqrt{t+L}]f(y) dy.
\]

Combining the average revenue, the average review cost, and the average inventory holding and backordering costs, the profit per unit time for the firm can be expressed as follows:

\[
E(t,z,p) = (p-c)D(p) - \frac{1}{\tau} \left( \frac{K}{t} + h\sigma D(p) \right) \int_{0}^{t} l(t,z,t) dt + b\sigma D(p) \int_{0}^{t} B(t,z,t) dt.
\]

The firm’s objective again is to maximize \( E \) by optimally selecting the review period \( t \), the safety stock factor \( z \) and the retail price \( p \).

3. Analysis of the TIB model

In this section, we analyze the behavior of the profit function for the TIB model—\( \Gamma(t,z,p) \) given in (2), in order to determine the relevant optimal decision variable values. First, we show that the maximization of \( \Gamma(t,z,p) \) can be reduced to a one-variable maximization problem in terms of \( t \). Then, we investigate the behavior of this one-variable function.

For any given review cycle length \( t \) and retail price \( p \), partial derivative of \( \Gamma \) with respect to \( z \) yields

\[
\frac{\partial \Gamma(t,z,p)}{\partial z} = \sigma D(p) \sqrt{t+L} \left\{ \frac{b}{t} \left[ 1 - F(z) \right] - h \right\}.
\]

The above expression means that the optimal safety stock factor is only related to \( t \) (independent of \( p \)). For any given \( (t,p) \), the unique \( z(t) \) maximizing \( \Gamma(t,z,p) \) is as specified in the following lemma.\(^{10}\)

**Lemma 1.** If \( t \geq b / 2h \), then \( z(t) = 0 \); otherwise, \( z(t) \) is the unique solution of \( 1 - F(z) = (b / h) t \). As a consequence of this, \( z(t) \) is decreasing\(^{11}\) for any \( t \in (0, b / 2h) \).

The above lemma clearly shows the effect of ignoring the time duration while determining the backordering cost in the approximate model, alluded to in Section 1. When \( t \) is low, the firm tries to reduce backordering costs by holding safety stock; however, as \( t \) increases, \( z(t) \), and hence the safety stock, decreases. Since the length for which backorders persist does not matter, when \( t \) becomes larger the firm’s optimal strategy is to focus more on controlling the holding cost through reduction of safety stock. In fact, when \( t \) is sufficiently large \( (\geq b / 2h) \), the firm does not hold any safety stock \( z(t) = 0 \). Obviously, the threshold \( t \) value (i.e., \( b / 2h \)) is higher if the per unit backordering cost is higher, and is lower if the cost of holding is higher.

The optimization of the original three-variable function \( \Gamma(t,z,p) \) can now be reduced to an optimization of a two-variable

\(^{9}\) The “small” number of backorders assumption implies that the holding cost in each review period can be calculated as the holding cost rate times the average stock level.

\(^{10}\) All proofs are provided in the Appendix.

\(^{11}\) Throughout the paper we use increasing/decreasing in the weak sense unless otherwise specified.
function $I(t,p)$ by substituting $z(t)$:

\[
I(t,p) = (p-c)(D(p) - \left\{ \frac{K}{t} + B(p) \right\} \mathcal{D}(p) \right) (z(t)).
\]

For notational convenience, let $B(t) = h/2 + b\alpha(t)\sqrt{t}/C0$ for any $t \in (0, +\infty)$. The above equation can then be rewritten as

\[
I(t,p) = \left( p - c \right) \mathcal{D}(p) - B(t)\mathcal{D}(p) - \frac{K}{t}.
\]

We can analyze the above profit function further to determine the optimal retail price $p$, in terms of $t$.

**Lemma 2.** For any given review cycle length $t$, $I(t,p)$ is unimodal in terms of $p$. So, the solution of $\partial I(t,p)/\partial p = 0$ provides the value of the optimal price $p(t)$. Specifically, for any $t \in (0, +\infty)$, we have

\[
p(t) = \frac{c - k_1}{1 + k_2} + \frac{1}{1 + k_2} B(t).
\]

Substituting the above $p(t)$ in $I(t,p)$ enables us to represent the firm’s profit function solely in terms of $t$, and we denote it by $I(t)$. In order to determine the optimal length of the review cycle, which is the maximizer of $I(t)$ over $(0, +\infty)$, we first characterize the behavior of $I(t)$ in terms of $t$ on $[b/2h, +\infty)$ (note that $B(t)$ has a unique minimizer for $t \in [b/2h, +\infty)$).

**Theorem 1.** Let $t_{\text{min}}$ be the unique minimizer of $B(t)$ on $[b/2h, +\infty)$, $t = \max\{b/2h, t_{\text{min}}\}$, and $t_0$ be the minimal value on $[t, +\infty)$ such that $G(t) = [B'(t)/B(t)]^2/[k_1 + k_2c + k_2b] + 1 > 0$. Then, there exists at most one local maximizer of $I(t)$ on $[b/2h, +\infty)$ and it is on $(t,t_0)$. Furthermore, $I(t)$ is quasi-concave in $(t,t_0)$.

Next, we focus on the behavior of $I(t)$ in terms of $t$ on $(0,b/2h)$. Differentiation of $I(t)$ results in

\[
\frac{dI(t)}{dt} = -B(t)D(p(t)) + \frac{K}{t^2}
\]

and

\[
\frac{d^2I(t)}{dt^2} = -B'(t)D(p(t)) - \frac{2K}{t^3}.
\]

From $dI(t)/dt = 0$, we get $B(t)D(p(t)) = K/t^2 > 0$. Hence, in order to characterize the shape of $I(t)$, it is crucial to understand the behavior of $B(t)$. So, in what follows, we study the behavior of $B(t)$ over $(0,b/2h)$. Before doing that, we first present a result about $r(z) = f(z)/(1-F(z))$ in terms of $z$ on $(0, +\infty)$ (recall that $F(z)$ and $f(z)$ are the density and cumulative distribution functions of $N(0, 1)$).

**Lemma 3.** Let $r(z) = f(z)/(1-F(z))$ and $g(z) = z(1-r(z))$. Then, $g(z)$ and $r(z)$ exhibit the following properties:

1. $0 < g(z) < 1$ for any $z \in (0, +\infty)$. Moreover, $g(z)$ is unimodal in terms of $z$ on $(0, +\infty)$.
2. $r(z)$ is strictly increasing and convex in terms of $z$ on $(0, +\infty)$.

Now we are ready to study the behavior of $B(t)$ over $(0,b/2h)$. As $B(t) = h/2 + b\alpha(t)\sqrt{t}/C0$ and $z(t)$ is the solution of $1-F(z) = h/2b\alpha(t)$ for any $t \in (0,b/2h)$, we have

\[
B(t) = \frac{h}{2} - b\alpha(t)\frac{t + 2L}{2\sqrt{t} + L} f(z(t)) - \frac{h\sqrt{t + L}}{b}\frac{t}{b - z(t)}.
\]

and

\[
B'(t) = \frac{h}{2} - b\alpha(t)\frac{t + 2L}{2\sqrt{t} + L} f(z(t)) - \frac{h\sqrt{t + L}}{b}\frac{t}{b - z(t)} f(z(t)).
\]

**Lemma 4.** $B'(t)$ is quasi-concave on $(0,b/2h)$. Hence, there exists at most one maximizer of $B(t)$ on $(0,b/2h)$.

Note that if $B(t) = 0$ has no solution in $(0,b/2h)$, the profit function would be increasing in $t$ for $t \in (0,b/2h)$. We can then focus on $(b/2h, +\infty)$ for the optimal $t$. On the other hand, if $B(t) = 0$ has solutions in $(0,b/2h)$, let $t_1$ be the smallest value and $t_2(> t_1)$ be the largest value in $(0,b/2h)$ such that $B(t) > 0$ on $(t_1,t_2)$. We can then focus our search only on $t \in (t_1,t_2)$ to determine the maximizer of $I(t)$ for $t \in (0,b/2h)$ as shown below.

**Theorem 2.** Any maximizer of $I(t)$ in terms of $t$ on $(0,b/2h)$ must be in $(t_1,t_2)$.

So, we can conclude that a simple one-dimensional search over $t \in (t_1,t_2)$ and/or $t \in (t_1,t_2)$ is enough for managers to select the optimal value of $t$. Substituting this $t$ in the corresponding $p(t)$ and $z(t)$ gives us the optimal values of all the three decision variables for the approximate model. We would like to point out that the characterization of the profit function of the TDB model as shown in Theorems 1 and 2 is one of our novel contributions to the literature.

Based on the above characterization, we can go further and analyze how the optimal decision variable values behave with respect to change in certain model parameters (i.e., sensitivity analysis). Note that we focus on the sensitivity analysis for two optimal decision variables—price ($p$) and review length ($t$). Sensitivity analysis for the optimal base stock level ($S$) is considerably more involved since it depends on $z$, $p$ and $t$. Consequently, that sensitivity analysis is done numerically in Section 5. For expositional convenience, we summarize the first order conditions for optimal price and review length below:

\[
p^* = \frac{c - k_1}{1 + k_2} + \frac{1}{1 + k_2} B(t^*)\frac{h}{2} + b\alpha(t^*)\sqrt{t^* - \frac{L}{t^*}} + \frac{h}{b}\frac{t}{b - z(t^*)} f(z(t^*)).
\]

\[
K = D(p^*)\left\{ \frac{h}{2} - b\alpha(t^*)\frac{t + 2L}{2\sqrt{t^*} + L} f(z(t^*)) - \frac{h\sqrt{t^* + L}}{b}\frac{t}{b - z(t^*)} f(z(t^*)) \right\}.
\]

**Proposition 1.** The following results about the change of the optimal decision variable values when certain model parameters are changed are true.

1. $\partial t^*/\partial b > 0$ if and only if $\partial p^*/\partial b > 0$.
2. $\partial t^*/\partial K > 0$ if and only if $\partial p^*/\partial K > 0$.
3. $\partial t^*/\partial c > 0$ if and only if $\partial p^*/\partial c > 0$.
4. $\partial t^*/\partial h > 0$ if and only if $\partial p^*/\partial h > 0$.

Two remarks are in order here regarding the above result. First, it shows that the behavior of the optimal price and the optimal review length are quite similar (i.e., either they both decrease or they both increase). The managerial intuition behind this result is as follows. If the review length increases, this increases the holding cost for the firm, but does not significantly change the backordering cost (recall that Proposition 1 is for TDB model). In order to counterbalance this cost increase, managers then need to increase the retail price (this will reduce the uncertainty level of the demand, hence the consequent reduction in safety stock also helps in reducing holding cost). On the other hand, if the review length decreases, managers can reduce the price, and increase the firm’s demand (as well as revenue), without significantly increasing the holding cost. As we show in Section 5, this relation between optimal review length and retail price is not necessarily true for the TDB model (e.g., in that case behavior of $p$ and $t$ with respect to $b$ might be totally opposite). Second, the above result implies that managers should be able to decide about how to change some of their decision variables as the business environment changes quite easily. For example, it is intuitive that the optimal review length ($t^*$) will always increase in the fixed review cost $K$. The above result
implies that managers should also increase the optimal retail price \( p^* \) as \( K \) increases. Similarly, since it is expected that \( p^* \) will increase in \( b \) (backordering cost per unit), the behavior of \( r' \) with respect to \( b \) then becomes evident for the managers.

4. Analysis of the TDB model

In this section we analyze the profit function for the TDB model—\( E(t_z, p) \) given by (4)—to determine the optimal decision variable values. First, for any given review cycle length \( t \) and retail price \( p \), differentiating \( E(t_z, p) \) with respect to \( z \) we get

\[
\frac{\partial E(t_z, p)}{\partial z} = \sigma D(p) \frac{\sqrt{t}}{t} \int_0^t \left[ b - (h + b)f(\delta) \right] dt. 
\]

Note that \( \int_0^t \left[ b - (h + b)f(\delta) \right] dt \) is strictly decreasing in terms of \( z \). If \( z \) tends to infinity, then \( \delta \) tends to infinity and \( f(\delta) \) tends to 1. This implies that \( \frac{\partial E(t_z, p)}{\partial z} < 0 \) when \( z \) is quite large. If \( z \) tends to zero, then \( \int_0^t \left[ b - (h + b)f(\delta) \right] dt \) tends to \( \int_0^t \left[ b - (h + b)f((t-t)/\sqrt{t+z+L}(1/\sigma)) \right] dt \). Therefore, we get the following result:

**Lemma 5.** There exists a unique positive constant \( t_0 \) such that for any \( t \in [t_0, +\infty) \), we have \( z(t) = 0 \). Otherwise, the unique \( z(t) \) satisfies \( \int_0^t \left[ b - (h + b)f(\delta) \right] dt = 0 \).

Recall that \( \delta = z \sqrt{t} \sqrt{1 + \frac{r^2}{2} / \sqrt{t + L + (t - t)/\sqrt{t + L}}} / \sqrt{t + L} \), (t - t)/\sqrt{t + L}) \), and it is strictly increasing in terms of \( z \). Hence, \( E(t, z, p) \) can be expressed in terms of \( (t, z, p) \) as follows (by substituting \( z(t) \)):

\[
E(t, p) = (p - c)D(p) \left\{ \frac{K}{T} + \frac{(h + b)}{D} \int_0^t \sqrt{t + L} f(\delta(t, \tau)) d\tau \right\}.
\]

For any given \( t \), taking partial derivative of \( E(t, p) \) with respect to \( b \) we get

\[
\frac{\partial E(t, p)}{\partial b} = D' \left\{ p + \frac{D}{D'} - c - \rho(t) \right\},
\]

where \( \rho(t) = (h + b)/(\sqrt{t} \sqrt{1 + \frac{r^2}{2} / \sqrt{t + L + (t - t)/\sqrt{t + L}}} / \sqrt{t + L}) \). Hence, there exists a unique \( p(t) \) such that \( E(t, p) \) is maximized and it satisfies \( p + D/D' = c + \rho(t) \), i.e.,

\[
p(t) = \frac{c - k_1}{1 + k_2} + \frac{\rho(t)}{1 + k_2}. \tag{11}
\]

Therefore, the maximization of \( E(t, p) \) can be reduced to a one-variable optimization problem in terms of \( t \) and this is summarized in the following theorem.

**Theorem 3.** For any given \( t \), there exist a unique \( z(t) \) and a unique \( p(t) \) such that \( E(t, z, p) \) is maximized. Hence, the maximization of \( E(t, p, z) \) can be reduced to a one-variable optimization problem in terms of \( t \).

Therefore, the TDB model involves maximization of the following function.

\[
E(t) = [p(t) - c]D(p(t)) - \frac{K}{T}D(p(t)).
\]

Differentiating \( E(t) \) with respect to \( t \), we get

\[
\frac{dE(t)}{dt} = \frac{K}{T} \rho(t)D(p(t))
\]

and

\[
\frac{d^2E(t)}{dt^2} = -\left\{ \frac{2K}{T} + \rho(t)D + \frac{1}{1 + k_2} \rho(t)^2 D \right\}.
\]

It is quite difficult to characterize \( E(t) = E(p(t)) \) in terms of \( t \). However, once again a one-dimensional search is enough to determine the optimal \( t \), and substituting this \( t \) in \( z(t) \) and \( p(t) \), we can determine the values of all the three decision variables for the TDB model.

Similar to what we did for the TIB model in Section 3, in the following, we would like to investigate the impact of some model parameters on the behavior of the optimal retail price \( (p^*) \) and the optimal review length \( (r') \). In this context, note that \( \rho(t) = (h + b)/t \int_0^t \sqrt{t + L} / \sqrt{t + L + (t - t)/\sqrt{t + L}} (1/\sigma) \). For convenience, we summarize the first order conditions for \( p^* \) and \( r' \) below:

\[
p^* = -\frac{k_1}{1 + k_2} + \frac{c + \rho(t^*)}{1 + k_2}
\]

and

\[
K = \rho(t^*)D(p^*).
\]

From the expressions of \( dE(t)/dt \) and \( d^2E(t)/dt^2 \), it is clear that we always have \( \rho(t^*) > 0 \) and \( 2K(t^*)^3 + \rho(t^*)D(p^*) + 1/(1 + k_2) \rho(t^*)^2 D(p^*) \geq 0 \). These two facts are useful in the following sensitivity analysis result.

**Proposition 2.** Let \( \rho_0 = k_1 + k_2 c + \rho(t^*) \), a constant. Then the following results about the change of the optimal decision variable values when certain model parameters are changed are true:

1. If \( \rho_0 < 0 \), then both \( \rho(t^*)/\rho_0 < 0 \) and \( \rho(t^*)/\rho_0 < 0 \).
2. If \( \rho_0 > 0 \), then both \( \rho(t^*)/\rho_0 < 0 \) and \( \rho(t^*)/\rho_0 > 0 \).
3. Both \( \rho(t^*)/\rho_0 > 0 \) and \( \rho(t^*)/\rho_0 > 0 \).
4. \( \rho(t^*)/\rho_0 > 0 \), then \( \rho(t^*)/\rho_0 > 0 \).

The important issue to note is that, for the TDB model, the behavior of the optimal review length with respect to the unit holding and backordering costs are quite similar. Specifically, if the holding and backordering costs are quite high (which makes \( \rho_0 \) high), then the optimal review length increases in \( h \) and \( b \); if these costs are relatively low (which makes \( \rho_0 \), relatively low), then the optimal review length decreases in \( h \) and \( b \). So, managers need to be careful about changing the review length as unit backordering or holding costs change. The intuition of the above result is as follows. When \( h \) and \( b \) are relatively low, any increase in either of them should be associated with a decrease in \( t' \) since this allows the firm to reduce the length of time for which backorders persist, and hence the backordering and holding costs (recall that Proposition 2 is for TDB model). As we will show in the next section, this similarity of behavior for \( t' \) with changes in \( h \) or \( b \) is not necessarily true for the TIB model.

With any increase of the fixed review cost \( K \), the optimal review cycle length \( r' \) should be decreased in order to reduce the average review cost. As a consequence of a longer \( r' \), the average holding cost is higher. In order to counterbalance this, the optimal retail price \( p^* \) should be increased (hence, the demand uncertainty level is lower and the safety stock level is also lower). Similarly, if \( r' \) is higher with higher \( \sigma \), then the optimal retail price \( p^* \) should be increased in order to balance the increased holding and backordering costs.

5. Comparison of the two models

Our above discussion indicates that TIB and TDB models are quite different in terms of how backordering cost is charged. We have also analytically characterized the behavior of the profit functions for both models showing that a one-dimensional search procedure would enable a manager to determine the optimal decision variable values and profits for each of them. Our (partial) analytical results regarding the impact of different model parameters on the optimal decision variables also show the distinct behavior of the two models. However, for both models...
the first-order-condition which determines optimal $t$ is rather involved. So, any further comparison of the optimal decision variables and profit values is analytically intractable. Keeping this in mind, we resort to extensive numerical experiments in order to demonstrate the difference in the optimal decisions/profits for the two models, and generate relevant managerial insights.

5.1. Numerical setting for comparison of the two models

For our numerical experiments, we consider two deterministic (price-sensitive) demand forms: iso-elastic for which $D(p)=ap^{-g}$ ($a>0, g>1$), and linear for which $D(p)=A-gp$ ($A>0, g>0$). These two are widely used to model price sensitive demands in the operations management literature (refer to Petruzzi and Dada, 1999; Wang et al., 2004; Granot and Yin, 2005). We focus on understanding the effects of the following parameters on the optimal decision values and the corresponding profits while comparing between the models: demand uncertainty ($\sigma$), leadtime ($L$), price-elasticity ($g$), review cost ($K$), backordering cost ($b$), and holding cost ($h$).

The base data set for the numerical study is as follows: $a=200$ and $g=1.5$ (for iso-elastic demand), $A=30$ and $g=2.0$ (for linear demand), $K=10$, $c=1.0$, $h=0.01$, $b=0.2$, $L=100$, $\sigma=3.0$ and $\mu=1$. In order to capture the effects of various parameters, we vary them (one at a time) around the base level. The three optimal values for the two models—review cycle length, retail price and base stock level—as well as the profits are then compared. We first discuss the comparison of the decisions and then the corresponding profits.

5.2. Optimal review cycle length

The effects of all the relevant parameters on the optimal review length are shown in Figs. 1 and 2. Notice that, in general, the optimal review cycle length for the TIB model is much higher than that for the TDB model. The underlying reason for this is as follows. Since there is no penalty in the TIB model for having backorders in the books for a long time (recall that the TIB model ignores the time duration of any backorder), this model increases the review length. In fact, such an action enables the firm to reduce the review and backordering costs (refer to $I'$ in (2)). Moreover, long review length does not significantly increase the safety stock holding costs. Note again from the expression of $I'$ in (2) that the rate of increase of safety stock holding costs is proportional only to the square root of the length of the review cycle $t$, while the backordering and review costs decrease as $t$ increases. The reason that the review cycle length is not even longer is that the cycle stock holding costs would then become very large (that portion of the costs increases linearly in $t$). However, in the TDB model too long review length is not good for the firm since it will result in lengthy backorders, and hence large backordering costs.

Interestingly, even the behavior of the optimal review length might be different in the two models. Based on Figs. 1 and 2, it is clear that the behavior of the optimal cycle length as $b$ changes is totally opposite in the two models. These phenomena are also related to the difference in how holding and backordering costs are charged in the two models. For example, as the value of $b$ becomes higher, they increase the potential cost of backorders. Since in the TDB model the backordering cost depends on how long backorders persist, the TDB model then tries to reduce the length of the review period (so that the backorders are in the books for relatively less time). On the other hand, length of backorders has no effect on the backordering cost for the TIB model. So, in that case we note that the model tries to increase the review length. When $\sigma$ is low, both models are similar to a deterministic one; hence, both their optimal review lengths are also quite close to that of the deterministic model.

Another interesting issue to note is that sometimes the TIB model can give rise to rather unreasonable results. As $K$ tends to zero, it is natural that the optimal $t$ should also tend towards zero. It is the case in the TIB model. However, for the TIB model, the optimal review length has a significantly large positive value even when the value of $K$ is almost zero. Again, this is because of the...
backordering cost structure in the TIB model. Note from the expression of $I^*$ in (2) that as $t$ tends to zero, the shortage tends to infinity. So, the TIB model will never make the review period length very small (and so, this model cannot approximate a continuous review model in the limiting case).

5.3. Optimal retail prices

The effects of the relevant parameters on the optimal retail price for each model are reported in Figs. 3 and 4. The behavior of the two prices in terms of each model parameter are similar, but the values again are quite different. In general, the optimal retail price of the TDB model is minor and the optimal decision values are close to the corresponding ones of the exact model. The optimal retail price is increasing in terms of each model parameter. The optimal decision variables in the TIB model, the intent of the model is to increase the review length and to decrease the retail price (which increases demand). This action allows it to reduce the review cost as well as increase revenue. But, the TDB model cannot do so since low price (and consequent high demand) will make the demand highly variable resulting in more (and lengthy) backorders (and so high backordering cost). The model then tries to generate revenue by charging high prices.\textsuperscript{15}

When $\sigma$ is low, the impact of the backordering cost structure in the TIB model is minor and the optimal decision values are close to the corresponding ones of the exact model. The optimal retail price is increasing in terms of both $\sigma$ and $L$ for both models. This can be explained in an intuitive way: the increase of $p$ offsets the increase in the demand variance as a result of higher $\sigma$ and $L$. For both models the optimal retail price $p$ is increasing in review cost $K$, the backordering cost rate $b$, and holding cost rate $h$, respectively. But, the optimal retail price is decreasing in terms of the price elasticity of the product and this is obvious: higher price sensitivity of the product, lower retail price to the customers.

5.4. Optimal base stock level

About the effects of the relevant model parameters on the optimal base stock level, we would like to point out that for all the test scenarios, we noted that the TIB model holds much lower safety stock than the TDB model. As explained before, the cost of backorders is underestimated in the TIB model (so the need for safety stock is less). The aim of the optimization in that case is to reduce the holding cost by keeping less safety stock. Obviously, the TDB model needs to keep higher safety stock in order to reduce backordering costs. This means that using the TIB model can indeed increase the risk of shortages in real-life. However, note that the optimal base stock level is determined by the following three factors: safety stock level, retail price, and the review length. Specifically, the base stock level can be calculated based on the following expression: $S = [(t+L) + \sigma \sqrt{(t+L)p}]$ (refer to (1)), where $\sigma \sqrt{(t+L)p}$ represents the safety stock part. Although $\sigma \sqrt{(t+L)p}$ part is significantly lower for the TIB model, $(t+L)p$ is actually higher for that model. This is so because the optimal review length is higher (refer to Section 5.2) and so is the demand (since optimal retail price is lower, refer to Section 5.3). Consequently, the overall optimal base stock level can be higher in either model.\textsuperscript{16} For example, as far as the effect of $\sigma$ is concerned, it seems that the optimal base stock level is dominated by the safety stock part for higher $\sigma$ values, while for lower $\sigma$ values it is the non-safety stock part which is more important.

5.5. Optimal profits

In order to understand the implications on profit as a result of using the TIB model if the backordering cost is time dependent, we proceed as follows. We first substitute the optimal review length, optimal retail price, and the optimal stocki...
factor values of the TIB model into the average profit function $E(t, p, z)$ of the TDB model (we denote this profit by $A^*$), and then compare $A^*$ with the optimal profit value $E^*$ of the TDB model ($E^*$ is obtained by substituting the three optimal decision variable values of the TDB model in $E(t, p, z)$). The effects of all the system parameters on the comparison are reported in Figs. 5 and 6.

It is obvious that $A^* < E^*$ for all scenarios. Before analyzing the models, our conjecture was that for most scenarios the absolute difference between $A^*$ and $E^*$ would be low (because the TIB model has been discussed and analyzed in the literature for a long time). In other words, the performance of the TIB model should be good enough under most business settings. In fact, this is not the case. Rather, in general, the penalty of using the optimal decisions from the TIB model is quite substantial for most market/operational settings. The only exceptions are the following scenarios—(i) when the demand uncertainty $\sigma$ is low (see Figs. 1–6), and (ii) when the total backordering cost is low, e.g., when both lead time $L$ and backordering cost rate $b$ are low (see Figs. 7 and 8).\(^{17}\) Only in those cases the optimal decision variable values for both models are quite similar and the differences between $A^*$ and $E^*$ are quite low (since the backordering costs are then relatively less important).\(^{18}\) Our analysis clearly shows that the review length plays a significant role in determining the expected profit of the TDB model, and any deviation from the true optimal can be harmful. More importantly, we establish that managers should be cautious about using the TIB model for their operations/pricing decisions if the backordering cost is time dependent. In most real-life settings such an action can be highly detrimental.

\[^{17}\] Note that in Figs. 7 and 8, we change both $b$ and $L$ simultaneously where $b$ is from 0.08 to 0.17 with step of 0.01 and $L$ is from 10 to 100 with step of 10.

\[^{18}\] Notice that if only $L$ or $b$ is low, then the differences between $A^*$ and $E^*$ are low, but the decision variable values for the two models can be quite different.
for the firm. Managers should be most careful when dealing with high-tech products or innovative ones or products in the incubation/growth phases of the lifecycle. Such products are normally characterized by high demand uncertainty and high backordering costs, e.g., fashion apparel, laptops, i-Pods, semiconductors, telecom equipments (Ray et al., 2005). Only for functional/commodity products and/or products in the maturity stage of the lifecycle, e.g., basic apparel, grocery, for which $b$ and $\sigma$ are low, using the TIB model might be acceptable.

General comment: We would like to point out that, although the values of the profits and decision variables for the two demand forms (iso-elastic and linear) are different, their behavior are almost the same for both models. In that sense, most of our above insights are quite robust.

6. Concluding remarks

In this paper we analyzed the joint operations-marketing decision-making for a firm dealing with an uncertain, price-sensitive environment and employing periodic review inventory management policy. The objective of the firm was to simultaneously decide on the review length of each period, the base stock level at each review time point, and the retail price of the product. We discussed two models—TIB (time-independent-backordering) and TDB (time-dependent-backordering)—in this context which differs mainly with regards to whether or not the length for which excess demands that are backordered persist are taken into consideration. Our main concern was to compare the performance of the two models under different business conditions, and provide managerial suggestions. First, we showed that the
optimization for both models can be reduced to one-variable problems. In this context, our characterization of the profit function of the TIB model is especially noteworthy because such analysis has not been done in the literature before. Second, we provided (partial) analytical sensitivity results for the two models, which indicated the difference in the behavior between the two models. Lastly, we carried out an extensive numerical study and showed that using the decision variables from the TIB model as approximations in the TDB model is not advisable for managers, irrespective of the form of the demand function. Specifically, because the TIB model underestimates the backordering costs, it tends to set longer-than-optimal review periods, charge lower-than-optimal retail prices and hold less-than-optimal safety stock. The differences in the decisions between the two models are usually quite substantial. Consequently, using the decisions from the TIB model results in significant profit penalty for most market and operating conditions. Only when the demand uncertainty in the market is quite low, using the TIB model might be an acceptable alternative. Keeping in mind the growing competition and volatility in the market (which implies high demand uncertainty and high backordering costs), we would strongly suggest that managers be extremely cautious about using the TIB model for decision-making in real-life in spite of its existence and popularity in the literature.

There are two possible extensions of this paper which are worthwhile endeavors. First, an extension to the random lead time model for decision-making in real-life in spite of its existence and operating conditions. Only when the demand uncertainty in the market is quite low, using the TIB model might be an acceptable alternative. Keeping in mind the growing competition and volatility in the market (which implies high demand uncertainty and high backordering costs), we would strongly suggest that managers be extremely cautious about using the TIB model for decision-making in real-life in spite of its existence and popularity in the literature.

Second, we extended our analysis to more general settings. For instance, we considered the case where the demand function is not necessarily linear. This extension is particularly relevant in the current economic environment, where volatile demand is the norm. Our results suggest that managers should be cautious when using the TIB model, especially in situations where demand uncertainty is high. In conclusion, our work provides valuable insights into the behavior of inventory models and highlights the importance of considering various factors in decision-making.
taking the first and second derivatives of $I(t)$ with respect to $t$, we get
\[
\frac{dI(t)}{dt} = -B(t)D(p(t)) + \frac{K}{t^2}
\]
and
\[
\frac{d^2 I(t)}{dt^2} = -B'(t)D(p(t)) - \frac{B(t)^2}{t^2} + \frac{2K}{t^3}.
\]
From $dI(t)/dt = 0$, we get $B(t)D(p(t)) = K/t^2 > 0$. Note that $B(t)$ is positive and convex since $B(t) = (h/2) - b(r(t+2L)/(2t^2 + \sqrt{t^2 + L}))$. Hence, near any maximizer of $I(t)$, $B(t)$ is strictly increasing and convex and $p(t)$ is also strictly increasing. Without loss of generality, in order to characterize the behavior of $I(t)$ on $[b/2h, \infty)$, we only need to focus on $t \in (\max\{b/2h, t_{\min}\}, \infty)$ where $t_{\min}$ is the minimizer of $B(t)$ on $[b/2h, \infty)$.

As $D(p)/D(p) = (k_1 + k_2)/((1+k_2)/(1+k_2) + k_2/(1+k_2))B(t)$, we get
\[
\frac{d^2 I(t)}{dt^2} = -B(t)D(p(t)) + \frac{B(t)^2}{t^2} + \frac{2K}{t^3}.
\]

The last equality is obtained as $D(p)/D(p) = k_1 + k_2B(t) = (1/1+k_2)/(1+k_2)B(t)$ and we need to analyze the behavior of $I(t)$ on $[b/2h, \infty)$. We divide the remaining proof into two cases based on the value of $k_2$.

1. **Case 1 ($k_2 \geq 0$):** In this case, it is obvious that $\{B(t)^2 + 2/tB(t)[k_1 + k_2 + k_2B(t)]\}$ is increasing in terms of $t$ (this can be verified easily by taking the first derivatives of both functions). Let $I(t) = [B(t)^2 + 2/tB(t)[k_1 + k_2 + k_2B(t)]]$ and we need to analyze the behavior of $I(t)$ on $[b/2h, \infty)$. We divide the remaining proof into two cases based on the value of $k_2$.

2. **Case 2 ($k_2 < 0$):** If we can show that $B(t)^2 + 2/tB(t)$ is also strictly decreasing, then $I(t)$ is increasing. Thus, it is sufficient to show that

\[
L(t) = \frac{B(t)^2 + 2}{B(t)}.
\]

is strictly decreasing. Based on the expression of $B(t)$, we can see that $B(t)$ is positive and increasing. On the other hand, we know that $B(t)$ is positive and increasing. Thus, $B(t)^2$ is decreasing. The remaining task is to show that $B(t)^2$ is strictly decreasing and this is obvious as

\[
\frac{B(t)^2}{B(t)^2} - 1 = b(t)0 = \frac{3t^2 + 4L}{2t^2 + \sqrt{t^2 + L}} \left( \frac{t + 2L}{2t^2 + \sqrt{t^2 + L}} \right)^{-1}.
\]

Finally, as both $B(t)^2$ and $B(t)$ are decreasing, $I(t)$ is strictly increasing.

Therefore, for both cases, $I(t)$ is strictly increasing. Let $t_0$ be the minimal value on $[\max\{b/2h, t_{\min}\}, \infty)$ such that $L(t) = (B(t)^2 + 2/tB(t)\{k_1 + k_2 + k_2B(t)\}) > 1$. Hence, $I(t)$ is quasi-concave on $[\max\{b/2h, t_{\min}\}, t_0]$ and quasi-convex on $[t_0, \infty)$. Note that both $B(t)$ and $p(t)$ tend to infinity while $t$ tends to infinity. Hence, $I(t)$ is non-positive while $t$ is very large and there does not exist any local maximizer of $I(t)$ on $(t_0, \infty)$. Regarding the behavior of $I(t)$ on $[b/2h, \max\{b/2h, t_{\min}\}]$, it is strictly increasing based on the expression of $I(t)$ and the convexity of $B(t)$ on $[b/2h, \infty)$. Thus, the local maximizer of $I(t)$, if any, must be in $[\max\{b/2h, t_{\min}\}, t_0]$ and $I(t)$ is quasi-concave on this interval. We complete the proof. □

**Proof of Lemma 3.** It is well known that $r(z)$ is strictly increasing (recall that our $f$ and $F$ represents standard normal). Hence $g(z) > 0$ for any $z \in (0, \infty)$. In order to show that $g(z) < 1$ for any $z \in (0, +\infty)$, it is equivalent to show that $z^2 - 2z(1 - F(z)) < 0$. As the left hand of this inequality tends to zero as $z$ tends to infinity, it is sufficient to show the first derivative of the left hand is positive, i.e., $2z(1 - F(z)) > 0$. This is obvious as $r' > 0$. Thus, $g(z) < 1$ for any $z \in (0, +\infty)$. Now we are ready to show the unimodality of $g(z)$. Note that $r' = r(r-z)$. Then, taking the first and the second derivatives with respect to $z$, we get

\[
g(z) = r - 2z + r(r-z)
\]

and

\[
g''(z) = 2r(r-z) - 2 + r(r-z)^2 + rz(r-z) - r = 2z(r-z) + r(2r-z) - r(z^2 - 1) < 0.
\]

The last inequality is because $g(z) < 1$ for any $z \in (0, \infty)$.

In the following, we only need to show that $r(z)$ is convex as $r' = (f(z) - z(1-F(z)))/(1-F(z))$. In order to show that $r' > 0$, it is equivalent to show that $f(z)(1-F(z)) - [f(z) - z(1-F(z))] > 0$. As the left hand tends to zero as $z$ tends to infinity, in order to show the above inequality, it is sufficient to show the first derivative of the left hand is negative, i.e.,

\[
A = -rf(z)^2 - x(1-F(z)) - zf(z)(1-F(z))
\]

\[
(1-F(z)) + f(z)(1-F(z)) < 0.
\]

Note that $A$ can be rewritten as

\[
A = (1-F(z))z(1-F(z)) - f(z)(1-F(z)) - zf(z)(1-F(z))
\]

\[
= \frac{\left[g(z)^2 \right]'}{(1-F(z)^2)}.
\]

Hence, it is sufficient to show that $g(z)$ is increasing. Note that $\lim_{z \rightarrow 0} g(z) = 0$, $\lim_{z \rightarrow \infty} g(z) = 1$, the value of 1 is the upper bound of $g(z)$, and $g(z)$ is unimodal in terms of $z$ on $(0, \infty)$, hence $g(z)$ is strictly increasing in terms of $z$ on $(0, \infty)$. □

**Proof of Lemma 4.** By rewriting the expression of $B(t)$ in (10), we get

\[
B(t) = \frac{b \sigma r}{\sqrt{t+L}(f(z) + 3t^2 + 4L + 8L^2)}
\]

\[
= \frac{\left(h^2 \left( 3t^2 + 4L + 8L^2 \right) \right)}{4t(t+L)^{3/2} f(z)(3t^2 + 12tL + 8L^2)^2}
\]

\[
= \frac{\left(h^2 \left( 3t^2 + 4L + 8L^2 \right) \right)}{4t(t+L)^{3/2} f(z)(3t^2 + 12tL + 8L^2)^2}
\]

where $r = f(z)/(1-F(z))$. The last equality is obtained because $(h/b)t = (h/c)$. The definition of $z$ again, by $1-F = (h/b)t$, we get

\[
z = \frac{1}{h} \frac{1}{b(1)} = - \frac{1}{t'}
\]

and $r' = r^2 - rz$

Taking the third derivative of $B(t)$ and utilizing the above two expressions, we get

\[
B''(t) = \frac{b \sigma \left( 3t^2 + 12tL + 8L^2 \right)}{4t(t+L)^{3/2} f(z)}
\]

\[
+ rz(2t^2 + 24L + 24L^2)
\]

\[
+ \left( h^2 \left( 3t^2 + 4L + 8L^2 \right) \right)
\]

\[
= \frac{\left(h^2 \left( 3t^2 + 4L + 8L^2 \right) \right)}{4t(t+L)^{3/2} f(z)(3t^2 + 12tL + 8L^2)^2}
\]

Finally, as both $B''(t)$ and $B(t)$ are decreasing, $I(t)$ is strictly increasing.
From $B'(t) = 0$, we get $z^2 = \left((3t^2 + 12tL + 8L^2)/4(t + L)(t + 2L)^2 - (t + L)/(t + 2L)\right)$. Substitution of these two expressions into $B''(t)$, we get

$$B''(t) = \left(\frac{2}{B}ight)^2 \frac{b\sigma}{4(t + L)^3/2} \left(2L^2 - 8t^2 + 8tL + 8L^2\right) - \frac{4t^2 - 4tL - 8L^2}{(t + L)^3}.$$ 

If $t \geq L$, it is easy to check that $B''(t) |_{B = 0} < 0$. If $t < L$, then $B'(t)$ can be rewritten as (note that $r = r^2 - zr$)

$$B''(t) = \left(\frac{2}{B}\right)^2 \frac{b\sigma}{4(t + L)^3/2} \left(2L^2 - 8t^2 + 16tL + 8L^2\right) - \frac{4t^2 - 4tL - 8t^2}{(t + L)^3}.$$ 

Taking the third derivative of $B(t)$, we get

$$B'''(t) |_{B = 0} = \left(\frac{2}{B}\right)^2 \frac{b\sigma}{4(t + L)^3/2} \left(2L^2 - 8t^2 + (10t + 12L)r - 8tL + 4(t + L)r - 8(t + L)L^3\right).$$

The last inequality is obtained since $r^* > 0$ and $z < 0$. From $B'(t) = 0$, we get $-tr^2 = -4(t + L)(t + 2L)/r^* + 4(t + L)^2$. Substitution of this expression into the right hand side of $B''(t) |_{B = 0}$, we get

$$B''(t) |_{B = 0} < \left(\frac{2}{B}\right)^2 \frac{b\sigma}{4(t + L)^3/2} \left(r^2 - 6L - 8L^2 + 4(t + L)L\right).$$

If $t \geq L$, then $B''(t) |_{B = 0} < (b/B)^2(\sigma/4(t + L)^3/2)(1/f^2(2)(2)(\tau)(t^2 - 6L - 8L^2 + 4(t + L)L) < 0$. □

**Proof of Proposition 1.** For simplicity, in this proof we denote the optimal decision variable values $(t', z')$ by $(t, p, z)$. We know that the optimal retail price and consumer cycle length $t$ satisfy $K/t = D(p)B(t)$ and $p = c_k(1/k + 1 + 1)(k^2 B(t),$ where $B(t) = (h/2)t + \sigma\sqrt{L}/t)f(z)$ and $B(t) = (h/2) - \sigma^2((t + 2L)/2(t^2 + \sqrt{L} + 2(t + 2L)/2)(t^2 + \sqrt{L} + 2(t + 2L)/2))t < 0$.

(1) We first consider the impact of $d$. Differentiating both sides of the above equation with respect to $b$, we get $(1 + k_2)\partial\sigma = \sigma\sqrt{L}/t)f(z) + B(t)\partial t/c$. From this fact, we know that if $\partial t/c > 0$, then $\partial p/c > 0$. On the other hand, differentiation on both sides of $B = D(p)B(t)$ with respect to $b$, we get

$$\frac{2K}{t^3} - DB''(t) \partial t/c = D'p \partial p/c - \sigma\sqrt{L}/t f(z).$$

Note that at the optimality, we must have $d^2 T(t)/dt^2 \leq 0$. Hence, we get $(-2K/t^2 - DB''(t) < 0$. Thus, if $\partial p/c > 0$, then we get $\partial t/c > 0$.

(2) Differentiating both sides of $p = (c_k + \sigma^2((t + 2L)/2(t^2 + \sqrt{L} + 2(t + 2L)/2))B(t)/c/k$ with respect to $K$, we get $\partial p/c = (1/k + 1)(k^2 B(t)\partial k/c$ (note that at the optimality we always have $B(t) > 0$). Hence, we see that $\partial t/c < 0$ if and only if $\partial p/c < 0$.

(3) Differentiating both sides of $K/t^2 = D(p)B(t)$ with respect to $s$, we get

$$\frac{2K}{t^3} - DB''(t) \partial s/c = D'p \partial p/c - \sigma\sqrt{L}/t f(z).$$

Hence, if $\partial t/c > 0$, then $\partial p/c > 0$.

On the other hand, differentiating both sides of $K/t^2 = D(p)B(t)$ with respect to $s$, we get

$$\frac{-2K}{t^3} - DB''(t) \partial s/c = D'p \partial p/c - \sigma\sqrt{L}/t f(z).$$

Hence, if $\partial p/c > 0$, then $\partial t/c < 0$.

(4) Differentiation on both sides of $p = (c_k + \sigma^2((t + 2L)/2(t^2 + \sqrt{L} + 2(t + 2L)/2))B(t)\partial t/c$ with respect to $L$ yields

$$\frac{-2K}{t^3} - DB''(t) \partial s/c = D'p \partial p/c - \sigma\sqrt{L}/t f(z).$$

Hence, if $\partial p/c > 0$, then $\partial t/c < 0$.

**Proof of Lemma 5.** Let $H(t) = \int_{b-(b+h)F(t+t)\sigma)/(\sigma\sqrt{L})}^t f(t + \sigma\sqrt{L}) dt$. First, we need to show that $H(t)$ is concave in terms of $t$ on $(0, +\infty)$. Differentiation with respect to $t$, we get

$$H'(t) = \frac{b-h}{\sigma\sqrt{L}} - \frac{(b+h)}{\sigma\sqrt{L}} \int_0^t f(t + \sigma\sqrt{L}) dt.$$ 

Note that $H'(t) < 0$. Hence, $H(t) < 0$, concave in terms of $t$ on $(0, +\infty)$. Also note that when $t$ tends to 0 from the right hand side of 0, we have $H(0^+) = 0$ and $H(0^-) = (b+h)/2 > 0$. Thus, there exists a positive $t_0$ such that $H(t) < 0$ for $t \in [t_0, +\infty)$ and $H(t) > 0$ for $t \in (0, t_0)$. We complete the proof. □

**Proof of Proposition 2.** For simplicity, in this proof we denote the optimal decision variable values $(t', z')$ by $(t, p, z)$. The two first order conditions are

$$p = \frac{c_k + \rho(t)}{1 + k^2} \quad \text{and} \quad \frac{K}{\sigma} = \rho(t)D(p).$$

(1) Differentiating both sides of the above two first order conditions with respect to $h$, we get

$$\frac{\partial p}{\partial h} = \frac{\rho'(t)\partial t}{\sigma} + \frac{1}{(1/k + 1)(k^2 B(t)\partial k/c},$$

and

$$\frac{-2K}{t^3} \partial h = \rho' D(c_k + \rho(t)\partial t/c + \rho D \partial p/c.$$

(2) Differentiating both sides of $p = (c_k + \rho(t))/1 + k^2$ with respect to $k$, we get $\partial p/c = (1/k + 1)(k^2 B(t)/c/k$ (note that at the optimality we always have $B(t) > 0$). Hence, we see that $\partial t/c < 0$ if and only if $\partial p/c < 0$.
Combining these two equations together, we get
\[
- \frac{2K}{t^2} - \rho^2 D - \frac{(\rho)^2 D}{1 + k_2} \frac{\partial t}{\partial K} = \frac{\rho^2 D}{(1 + k_2)b + h_2}. \]

From this, we can see that the results hold true. Similarly, we can show the results with respect to \( h \). We complete the proof for parts (1) and (2).

(3) Differentiating both sides of the first two order conditions with respect to \( K \), we get
\[
- \frac{2K}{t^2} - \rho^2 D - \frac{(\rho)^2 D}{1 + k_2} \frac{\partial t}{\partial K} = - \frac{1}{t^2} + \rho^2 D \frac{\partial \rho}{\partial K},
\]
and
\[
\frac{\partial p}{\partial K} = \rho^2 \frac{\partial t}{1 + k_2}. \]
Combining them together, we get
\[
- \frac{2K}{t^2} - \rho^2 D - \frac{(\rho)^2 D}{1 + k_2} \frac{\partial t}{\partial K} = - \frac{1}{t^2}. \]

Hence, we see that \( \partial t/\partial K > 0 \) and \( \partial p/\partial K > 0 \).

(4) Differentiating both sides of \( p = -(k_1/(1 + k_2)) + (c + \rho(t))/ (1 + k_2) \) with respect to \( \sigma \), we get
\[
\frac{\partial p}{\partial \sigma} = \frac{1}{1 + k_2} \frac{\partial t}{\partial \sigma} + \frac{\rho^2}{t} \int_0^t \sqrt{\tau + hf(t, \tau)} d\tau + \frac{(\rho + b)\sigma}{t} \int_0^t f(t, \tau) h(t, \tau) t^2 \sigma d\tau.
\]

Thus, it is obvious that if \( \partial t^2/\partial \sigma > 0 \), then \( \partial p^2/\partial \sigma > 0 \).

References


